Probability and Random Processes

R.J. Marks II Lecture Notes

University of Washington (1984)

TIE PROGRAM

Summer Quarter 1984

EE 505 - Introduction to Probability and Random Processes
4 credits

Professor Robert Marks

12:00 - 1:00 M, Th
12:00 - 2:10 T

List of students in the Televised Instruction in Engineering Program who are enrolled in this course, their telephone number, and their company affiliation.

1.	Baruah, Arati Bora	Boeing Computer	O:	656-5741
			H:	271-1773
2.	Brown, Marc Gerard	Hewlett-Packard	O:	335-2028
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3.	Chastain, Paul Gene	John Fluke	O:	356-5281
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4.	Corulli, Charles	Fairchild	O:	841-6022
			H:	858-7075
5.	Ellersick, Steven Donald	Boeing Military	O:	394-4268
			H:	324-9125
6.	Fitzsimmons, Dan Kenneth	Boeing Aerospace	O:	251-0238
			H:	838-7185
7.	Lavering, Laura Jean	Boeing Computer	O:	
			H:	
~~8.~~	~~Nolan, Jim~~ Decided on ME411	~~Puget Power~~	~~O:~~	~~882-4441~~
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9.	Nguyen, Le Huu	Boeing Aerospace	O:	655-1697
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			H:	392-8156

Solutions

EE 505
Midterm

INSTRUCTIONS:

- Monday, July 15, 1996; 2:20 PM to 4:20 PM.
- Write your name on the upper right hand side of this sheet.
- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed a single legal sized sheet of notes and calculator.
- Each problem is worth 20 points.
- TIE students must identify the exam proctor and have the proctor initial the examination.

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1. Specify whether the following pairs of events are (a) independent, (b) mututally exclusive or (c) neither[1].

- A balanced budget amendment bill passes congress by August 15. A balanced budget amendment does not pass congress by August 15.
  → mutually exclusive
- The sum on two dice is seven. There are six dots on the first die. → Ind.
  $Pr[\text{sum } 7, 6] = \frac{1}{36}$ = PRODUCT OF TWO PROBABILITIES.
- You have an ace in your poker hand. Your opponent has an ace in their poker hand. Neither
- You win the Washington state lottery. Your mother wins the New York lottery. Ind.
- You receive one call before noon. You receive two calls all day. neither

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[1] Four points for a correct answer, zero for no answer and -2 for an incorrect answer.

2. Ken Griffey Jr. has a batting average of 0.300. Assume this means, each time he bats, his probability of getting a hit is 0.300. Estimate the probability that he gets 300 or more hits in his next 1000 at bats.

$p = 300 \; ; \; n = 1000$

$$\Pr[k \geq 300] = \sum_{k=300}^{1000} \binom{n}{k} p^k q^{n-k}$$

$$npq = .300 \times 1000 \times 0.70 = 210$$

$$\begin{aligned} &\Pr[k \geq 300] \\ &= \Pr[k < 300] \approx \mathcal{G}\left(\frac{300 - np}{\sqrt{npq}}\right) \\ &= \mathcal{G}(0) = \frac{1}{2} \end{aligned}$$

3. In Lake Wobegon, there are 10,000 catfish, 20,000 perch and 30,000 dogfish. The dogfish are very hungry and are twice as likely to be caught. Bobby Jack went fishing and caught four fish. The probability that three were dogfish and one was a perch were caught can be written as

$$\frac{2^P}{3^Q}.$$

What are the integers P and Q?

CATFISH(1) PERCH(2) DOGFISH(3)

$$p_1 = \frac{10^k}{10k + 20k + 2 \times 30k}$$

$$p_1 = \frac{1}{9} \qquad p_2 = \frac{2}{9} \qquad p_3 = \frac{6}{9} = \frac{2}{3}$$

$$k_1 = 0 \qquad k_2 = 1 \qquad k_3 = 3$$

Generalized Bernoulli trial:

$$P_r[k_1\ k_2\ k_3] = \frac{n!}{k_1!\,k_2!\,k_3!}\, p_1^{k_1}\, p_2^{k_2}\, p_3^{k_3}$$

$$P_r[0, 1, 3] = \frac{4!}{0!\,1!\,3!} \left(\frac{1}{9}\right)^0 \left(\frac{2}{9}\right)^1 \left(\frac{2}{3}\right)^3$$

$$= 4 \cdot \frac{2}{3^2} \cdot \frac{2^3}{3^3} = \frac{2^6}{3^5}$$

$$P = 6, \quad Q = 5$$

$$\left(\frac{2^6}{3^5} = 0.263\right)$$

4. Bill eats only Big Macs and sausage pizzas. Bic Macs give him heartburn 10% of the time. The pizza's give him heartburn 20% of the time. He eats twice as many pizzas as Big Macs. Bill has heartburn. What is the probability it was caused by a Big Mac?

$$P[H] = P[H/BM]\,P[BM] + P[H/Pizza]\,P[Pizza]$$

$$= \frac{1}{10} \times \frac{1}{3} + \frac{2}{10} \times \frac{2}{3} = \frac{5}{30} = \frac{1}{6}$$

$$P_r[BM/H] = \frac{P_r[H/BM]\,P_r[BM]}{P[H]}$$

$$= \frac{\frac{1}{10} \times \frac{1}{3}}{1/6} = \frac{6}{30} = \frac{1}{5} = 0.20$$

5. A Poisson random variable with parameter $\lambda = 2$ occurances per hour is observed for a half hour. The probability that the number of occurances exceeds or is equal to two *given* that the total number of occurances exceeds or equals one can be written as

$$\frac{1 - a\,e^{c}}{1 - b\,e^{d}}.$$

Identify the nu [illegible], b, c and d.

$$\lambda T = 2 * \tfrac{1}{2} = 1$$

$$\wp = \text{Prob}[\,X \geq 2 \mid X \geq 1\,] = \frac{P_r[X \geq 2 \text{ and } X \geq 1]}{P_r[X \geq 1]}$$

$$= \frac{P_r[X \geq 2]}{P_r[X \geq 1]}$$

$$= \frac{1 - [\text{Prob}(X=0) + \text{Prob}(X=1)]}{1 - \text{Prob}(X=0)}$$

$$P_r[X = k] = \frac{(\lambda T)^k}{k!} e^{-\lambda T} = \frac{1^k}{k!} e^{-1} = \frac{e^{-1}}{k!}$$

Thus

$$P_r[X=0] = e^{-1} \quad ; \quad P_r[X=1] = e^{-1}.$$

and

$$\wp = \frac{1 - (e^{-1} + e^{-1})}{1 - e^{-1}} = \frac{1 - 2e^{-1}}{1 - e^{-1}}$$

Thus: $a = 2$, $b = c = d = 1$.

6. Washington state apples are modeled with a Gaussian pdf. If X is the diameter,

$$X \sim N(\mu = 3, \sigma = 2)$$

Apples below two inches in diameter and above four inches are discarded. What is the probability that an apple passing this test is three inches or less in diameter?

$$P = \Pr[X \le 3 \mid 2 \le X \le 4]$$

$$= \frac{\Pr[X \le 3, \ 2 \le X \le 4]}{\Pr[2 \le X \le 4]}$$

$$= \frac{\Pr[2 \le X \le 3]}{\Pr[2 \le X \le 4]}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Recall:

$$\Pr[x_1 \le X \le x_2] = \mathbb{G}\left(\frac{x_2-\mu}{\sigma}\right) - \mathbb{G}\left(\frac{x_1-\mu}{\sigma}\right)$$

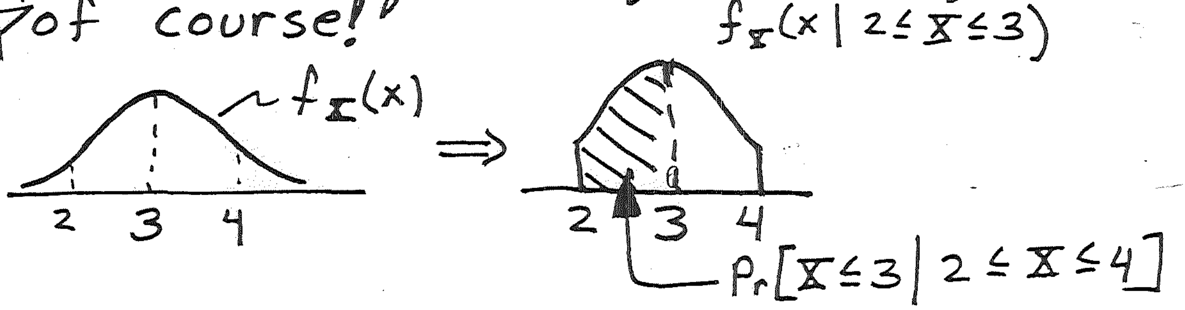
$$= \operatorname{erf}\left(\frac{x_2-\mu}{\sigma}\right) - \operatorname{erf}\left(\frac{x_1-\mu}{\sigma}\right)$$

Thus

$$P = \frac{\operatorname{erf}((3-3)/2) - \operatorname{erf}\left(\frac{2-3}{2}\right)}{\operatorname{erf}\left(\frac{4-3}{2}\right) - \operatorname{erf}\left(\frac{2-3}{2}\right)}$$

$$= \frac{\operatorname{erf}(0) - \operatorname{erf}(\frac{-1}{2})}{\operatorname{erf}(\frac{1}{2}) - \operatorname{erf}(\frac{-1}{2})} = \frac{\operatorname{erf}(\frac{1}{2})}{2\operatorname{erf}(\frac{1}{2})} = \frac{1}{2}$$

☆ of course!



7. Matlab's error function is

$$\operatorname{erf}_{ML}(x) = \frac{2}{\sqrt{\pi}} \int_{t=0}^{x} e^{-t^2}\, dt$$

Papoulis' definition is

$$\operatorname{erf}(y) = \frac{1}{\sqrt{2\pi}} \int_{z=0}^{y} e^{\frac{-z^2}{2}}\, dz$$

We wish to find erf(2) using Matlab. How do you do it?

$$\operatorname{erf}(y) = \frac{1}{\sqrt{2\pi}} \int_{0}^{y} e^{-z^2/2}\, dz$$

$$t = z/\sqrt{2} \Rightarrow dz = \sqrt{2}\, dt$$

when $z = y$, $t = y/\sqrt{2}$. Thus

$$\operatorname{erf}(y) = \frac{1}{\sqrt{2\pi}} \int_{0}^{y/\sqrt{2}} e^{-t^2} (\sqrt{2}\, dt)$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{y/\sqrt{2}} e^{-t^2} dt$$

$$= \frac{1}{2} \left[\frac{2}{\sqrt{\pi}} \int_{0}^{y/\sqrt{2}} e^{-t^2} dt \right]$$

$$= \frac{1}{2} \operatorname{erf}_{ML}\left(y/\sqrt{2} \right)$$

Thus:

$$\operatorname{erf}(2) = \frac{1}{2} \operatorname{erf}_{ML} \frac{2}{\sqrt{2}} = \frac{1}{2} \operatorname{erf}_{ML}(\sqrt{2})$$

Solution

EE 505
Midterm
Monday, July 25, 1997
2:20 pm to 4:30

INSTRUCTIONS:

- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed a single legal sized sheet of notes and calculator.
- Each problem is worth 20 points.
- *The Error Function Table is on Page 8 of this booklet.*

---

1. Specify whether the following pairs of events are (a) independent, (b) mutually exclusive or (c) neither[1].

- A health care $\overset{A}{\text{bill}}$ passes congress by August 15. A health $\overset{B}{\text{care}}$ bill does not pass congress by August 15. (B)
- The sum on two dice is seven. There are six dots on the first die. (A)
- You have an ace in your poker hand. Your opponent has an ace in their poker hand. (C)
- You win the $\overset{A}{\text{Washington}}$ state lottery. Your mother $\overset{B}{\text{wins}}$ the New York lottery. (a)
- You receive one call before noon. You receive two calls all day. (C)

1. Let $A = \{\text{bill passes}\}$, $B = \{\text{bill does not pass}\}$
Then $B = A^c$ $\therefore$ Mutually exclusive (b)

2. $P[\text{Sum is } 7] = \frac{1}{6} = P[A]$
$P[6 \text{ on 1st die}] = \frac{1}{6} = P[B]$
$P[\text{Sum is } 7 \ \& \ 6 \text{ on 1st die}] = P[(6,1)] = \frac{1}{36} = P[A \cdot B]$
$= P[A] \cdot P[B]$ $\therefore$ Independent (a)

3. $P[A] = 4/52$, $P[B] = 4/52$, $P[AB] = P[A|B]P[B]$
$= 3/51 \cdot 4/52 \neq P[A]P[B]$ & $P[AB] \neq 0$ $\therefore$ (c)

4. $P[AB] = P[A]P[B] \Rightarrow$ (a)

5. $P[AB] \neq P[A]P[B]$, $P[AB] \neq 0$ $\therefore$ (c)

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[1] Four points for a correct answer, zero for no answer and -2 for an incorrect answer.

2. Ken Griffey Jr. has a batting average of 0.300. Assume this means, each time he bats, his probability of getting a hit is 0.300. Estimate the probability that he gets 850 or more hits in his next 1000 at bats.

$p = 0.3, \quad q = 0.7, \quad n = 1000, \quad np = 300$

$p[850 \text{ or more in } 1000] = p[850 \leq k \leq 1000]$

$$= \sum_{k=850}^{1000} \binom{n}{k} 0.3^{k} \, 0.7^{n-k}$$

$$= G\left(\frac{1000 - np}{\sqrt{npq}}\right) - G\left(\frac{850 - np}{\sqrt{npq}}\right)$$

$$= G\left(\frac{700}{14.5}\right) - G\left(\frac{550}{14.5}\right)$$

$$= G(48.2) - G(37.9)$$

$$\simeq 0$$

3. In Lake Wobegon, there are 10,000 catfish, 20,000 perch and 30,000 dogfish. The dogfish are very hungry and are twice as likely to be caught. Bobby Jack went fishing and caught four fish. What is the probability that three were dogfish and one was a perch?

Let

$A = \{ \text{catch Dogfish} \}$,

$B = \{ \text{catch perch} \}$,

$C = \{ \text{Catch catfish} \}$

Then

$$p[A] = \frac{6}{9} = p$$

$$p[B] = \frac{2}{9} = q$$

$$p[C] = \frac{1}{9} = r$$

$$P_4(3,1) = \frac{4!}{3!\,1!} \left(\frac{6}{9}\right)^3 \left(\frac{2}{9}\right)^1$$

$$\simeq 0.26$$

4. Bill eats only Big Macs and sausage pizzas. Bic Macs give him heartburn 10% of the time. The pizza's give him heartburn 20% of the time. He eats twice as many pizzas as Big Macs. Bill has heartburn. What is the probability it was caused by a Big Mac?

HB = HeartBurn, BigMac = BM, pizza = PI

$$p(HB)$$

$$= p(HB, BM) + p(HB, PI)$$

$$= \underbrace{p(HB|BM)}_{0.1}\, \underbrace{p(BM)}_{\frac{1}{3}} + \underbrace{p(HB|PI)}_{0.2}\, \underbrace{p(PI)}_{2/3}$$

$$= 0.1 \times \frac{1}{3} + 0.2 \times \frac{2}{3}$$

$$= \frac{0.5}{3}$$

$$p(BM|HB) = \frac{p(HB, BM)}{p(HB)}$$

$$= \frac{\frac{0.1}{3}}{\frac{0.5}{3}}$$

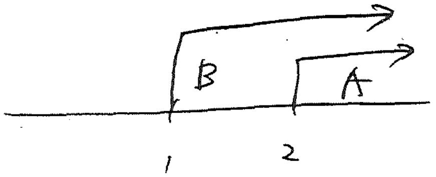
$$= \frac{1}{5} \,//$$

5. A Poisson random variable with parameter $\lambda = 2$ occurrences per hour is observed for a half hour. What is the probability that the number of occurrences exceeds two *given* that the total number of occurrences exceeds one?

Let $A = \{\#\text{ of occurrences} > 2\}$

$B = \{ \quad 〃 \quad > 1\}$

Clearly $B \supseteq A$



$$
\begin{aligned}
p\{A\} &= p[k > 2] \\
&= 1 - p[0 \le k \le 2] \\
&= 1 - \{p[k=0] + p[k=1] + p[k=2]\} \\
&= 1 - \{e^{-1}\left(\frac{1^0}{0!} + \frac{1^1}{1!} + \frac{1^2}{2!}\right)\} \\
&\simeq 0.08
\end{aligned}
$$

$$
\begin{aligned}
p\{B\} &= p[k > 1] \\
&= 1 - p[k=0] - p[k=1] \\
&= 1 - [e^{-1}(1) + e^{-1}(1)] \\
&\cong 0.26
\end{aligned}
$$

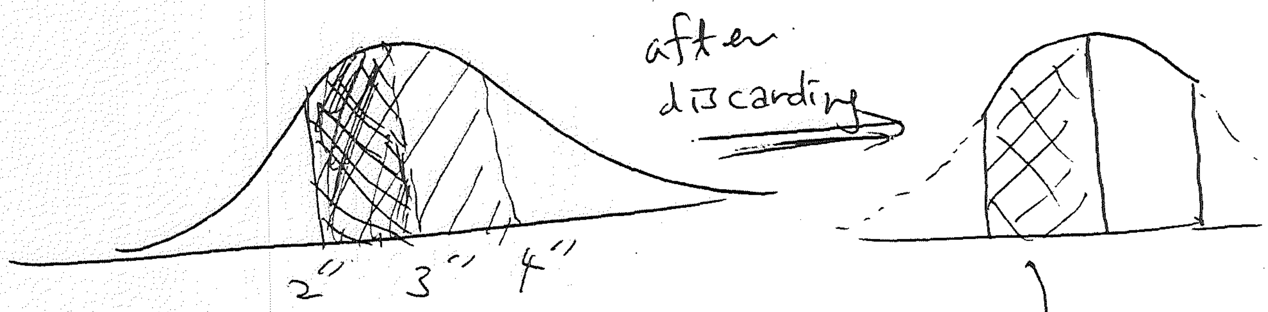
$$
p\{k > 2 \mid k > 1] = \frac{p[A, B]}{p[B]} = \frac{p[A]}{p[B]} = \frac{0.08}{0.26}
$$

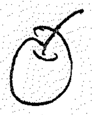
$$\simeq 0.30$$

6. Washington state apples are modeled with a Gaussian pdf. If X is the diameter,

$$X \sim N(\eta = 3, \sigma = 1)$$

Apples below two inches in diameter and above four inches are discarded. What is the probability that an apple, after the discarding, is three inches or less in diameter?



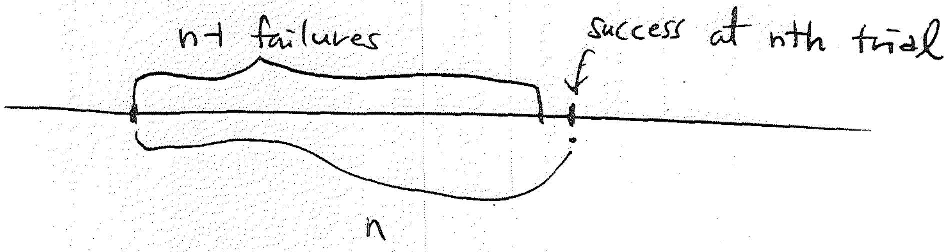
P[ ≤ 3 | Discarding]

$= \frac{1}{2}$

7. Let Y be a Bernoulli trial with probability of success p. We perform the Bernoulli trial until we get a success. Let N denote the number of trials needed to achieve a success. What is the pdf of N?

We just need one success. Then we stop.

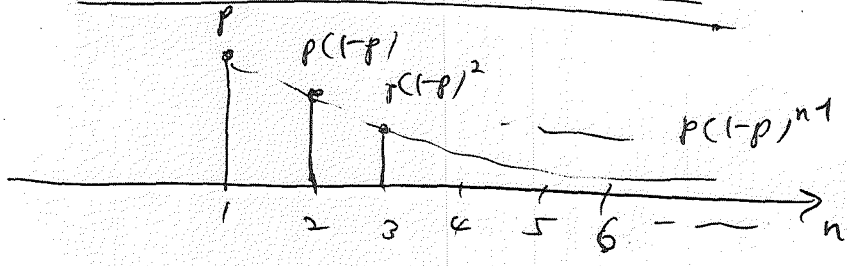
if n is # of trials, this means



$$\Downarrow$$

$$f_N(n) = \binom{n-1}{0} p^0 q^{n-1} \times p$$

$$= p \cdot (1-p)^{n-1}, \quad n \geq 1$$



check: $$\sum_{n=1}^{\infty} f_N(n) = 1$$

$$\sum_{n=1}^{\infty} p(1-p)^{n-1} = p \sum_{n=1}^{\infty} (1-p)^{n-1}$$

$$= p \cdot \frac{1}{1-(1-p)} = \frac{p}{p} = 1$$

SOME CORRECTIONS ON EE505 TEXT (PAPOULIS)

1. Page 82, Ea. (4-49)

$$\frac{ce^{-cx}}{ce^{-ct}} = e^{-c(x-t)} \quad \text{should be} \quad \frac{ce^{-cx}}{e^{-ct}} = ce^{-c(x-t)}$$

2. Page 75, Eq. (4-32)

$\delta(n-k)$ should be $\delta(x-k)$

3. Page 92, Ex. 5-3, second sentence

$F_x(-c) - F_x(c)$ should be $F_x(c) - F_x(-c)$

4. Page 148, Problem 6-18

Mising right paren

5. Page 171, Problem 7-10

$E\{U(a-x)\} E \{U(a-y)\}$ should be $E\{U(a-x)\} E \{U(b-y)\}$

6. Page 132, Two lines above second equation

$x < b \cos \theta$ should be $x < a \cos \theta$

7. Page 132, Second equation

Upper integration limit should be $\pi/2$ instead of a

8. Page 195, First equation

$\eta_i = T$ should be $T/2$

Next line, $\eta = 2T$ should be $\eta = T$

9. Page 210, Third equation.

   The two cosine terms should be added, not subtracted

10. Page 213, Eq. 9-20

   2^k should be 2^n

11. Page 232, Third equation

   The product $x_1 x_2$ should be deleted

12. Page 224, Last equation

   The $n=0$ and $n \neq 0$ should be interchanged

13. Page 252, Last equation

   Bracket and semicolon are missing

14. Page 259, Problem 9-4, third line

   y in y(t) should be bold faced

15. Page 261, Problem 9-30

   "...where $x(t)=$" should be "...where $R_x(\tau)=$"

16. Page 572, Index

   Gamma density entry should list page 77

17. Page 9, Example 1-3, third line

$\ell\sqrt{3}$ should be $r\sqrt{3}$

18. Page 60, last line before problems

r should be p_r

19. Page 76, second line

46 should 40

20. Page 93, Ex. 5-6, second equation

missing right paren

21. Page 100, one line after 5-14

x should be y

22. Page 122, problem 5-27

$2q/p^2$ should be q/p^2

23. Page $1\underset{7}{2}0$, Fig. $6\text{-}5^b$

x should be x_3

24. Page 141, Ex. 6-10, first equation

second $x(f_i f_k)$ should be $y(f_i f_k)$

25. Page 154, Ex. 7-2

$$r_{zw}^2 = \frac{[E(z\overset{w}{\cancel{a}})-E(z)E(w)]^2}{\sigma_z^2\ \sigma_w^2} = \frac{9}{7\text{x}3}$$

26. Page 154, Ex. 7-2, Last equation should be

$$N(10,10;\sqrt{7},\sqrt{3};\sqrt{3/7})$$

27. Page 171, Problem 7-15

$$E\{[y-g(x)]^2\}$$

28. Page 187, last equation

missing $d\chi$

29. Page 192

(a) line above fourth equation

$p(1-q)$ should be $p(1-p)$

(b) fourth equation

second inequality should be reversed

30. Page 198, first equation

denominator should be $\sqrt{n}\,\sigma$ instead of σ

31. Page 212, line above Eq. (9-19)

"of" should be "or"

32. Page 21~~3~~4, Eq. 9-24

$f(w,t) =$

33. Page 260, Probl. 9-16

$$G\left(\frac{a}{\sqrt{-R_{XX}(0)}}\right)$$

34. Page 266, Eq. 10-13

delete $1/2\pi$

Acknowledgements: 17-33 were detected by Peter Wai

WORK

(3-1)

② → $(p+q)^n - (p-q)^n \Leftarrow$ GIVES even

③ →

$\binom{K}{k}$ = # orderings k good

$\binom{N-K}{n-k}$ = # " n-k bad

$\binom{K}{k}\binom{N-K}{n-k} \Leftarrow$ # k Good & n-k bad

4. $n = 900$ $np = 450$ $\sigma^2 = npq = \frac{900}{4} = 225$

$420 < k < 465$ $\sigma = 15$ $\frac{4}{0}$

$\approx \cancel{\sum_{420}}$ $\operatorname{erf}\left(\frac{k_2 - np}{\sigma}\right) - \operatorname{erf}\left(\frac{k_1 - np}{\sigma}\right)$

$= \operatorname{erf}\left(420 - \right.$

7. $p = 1 - e^{-T/4/T} = 1 - e^{-\frac{1}{4}}$

$npq >> 1 \Leftarrow$ DeMoirve npq

$\int_0^{800}$

~~12~~ ⑧ k-1 heads and (n-k) tails at n-1

$\binom{n-1}{k-1} p^{k-1} q^{n-k\cancel{(k-1)}} \times p \leftarrow n^{TH}$ TIME

⑭ Russian Roulette

$P(A) = P(A/\mu)\,P(\mu) + P(A/\bar{\mu})\,P(\bar{\mu})$

$p = 1 \times \frac{2}{36}$ ~~P~~ + $(1-p)\,\cancel{8}\,\frac{34}{36}$

$p\left(1 + \frac{34}{36}\right) = \cancel{\frac{2}{32}}\ 1$

$p\ \frac{70}{36} = \cancel{2/36}\ \frac{36}{70} = \frac{18}{35}$

$\cancel{2/70}$

1. For $Y = \frac{1}{N}\sum_{n=1}^{N} X_n$

<u>if</u> all the X_n's are independent:

$$\Phi_Y(\omega) = \Phi_X^N\left(\frac{\omega}{N}\right)$$

Thus:

$$j\overline{y} = \frac{d}{d\omega}\Phi_Y(0)$$

$$\frac{d}{d\omega}\Phi_X^N\left(\frac{\omega}{N}\right) = \frac{1}{N}\, N\, \Phi_x^{N-1}\left(\frac{\omega}{N}\right)\Phi_x'\left(\frac{\omega}{N}\right) = \frac{d}{d\omega}\Phi_Y(\omega)$$

$$\Rightarrow j\overline{y} = \Phi_X^{N-1}(0)\,\Phi_x'(0) = 1^{N-1}\cdot j\overline{x} \Rightarrow \overline{y} = \overline{x}$$

Now: $-\overline{y^2} = \left(\frac{d}{d\omega}\right)^2\Phi_Y(0)$

$$\left(\frac{d}{d\omega}\right)^2\Phi_Y(\omega) = \frac{d}{d\omega}\Phi_X^{N-1}\left(\frac{\omega}{N}\right)\Phi_x'\left(\frac{\omega}{N}\right)$$

$$= \frac{N-1}{N}\Phi_X^{N-2}\left(\frac{\omega}{N}\right)\left[\Phi_x'\left(\frac{\omega}{N}\right)\right]^2 + \frac{1}{N}\Phi_X^{N-1}\left(\frac{\omega}{N}\right)\Phi_x''\left(\frac{\omega}{N}\right)$$

$$-\overline{y^2} = \frac{N-1}{N}\Phi_X^{N-2}(0)\left[\Phi_x'(0)\right]^2 + \frac{1}{N}\Phi_X^{N-1}(0)\,\Phi_X''(0)$$

$$= \frac{N-1}{N}(1)(j\overline{x})^2 + \frac{1}{N}(1)(-\overline{x^2})$$

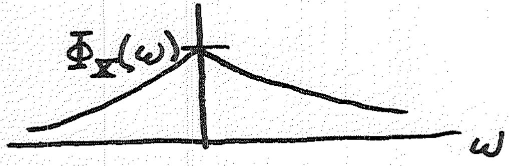
$$\overline{y^2} = \frac{N-1}{N}\overline{x}^2 + \frac{1}{N}\overline{x^2}$$

$$\Rightarrow \text{var}\,Y = \overline{y^2} - \overline{y}^2 = \frac{N-1}{N}\overline{x}^2 + \frac{1}{N}\overline{x^2} - \overline{x}^2$$

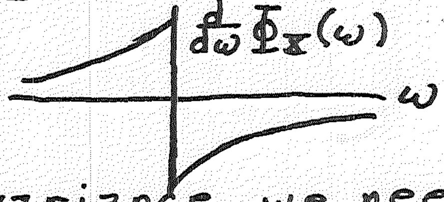
$$= \frac{1}{N}\left[\overline{x^2} - \overline{x}^2\right] = \frac{1}{N}\text{var}\,X$$

2. For Cauchy: $f_X(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}$

From last homework: $\Phi_X(\omega) = e^{-\alpha|\omega|}$



$j\overline{x} = \frac{d}{d\omega}\Phi_X(0)$. But $\frac{d}{d\omega}\Phi_X(\omega)$ looks like this:



$\Rightarrow$ so $j\overline{x} = ?$
Not defined

To get variance, we need $-\overline{x^2} = \left(\frac{d}{d\omega}\right)^2\Phi_X(0)$. But $\left(\frac{d}{d\omega}\right)^2\Phi_X(\omega)$ has a delta function at the origin and $\overline{x^2} = \infty$.

- Find the density, mean and variance of

$$\underline{Y} = \frac{1}{N}\sum_{n=1}^{N} \underline{X}_n$$

when $\underline{X}$ is (a) Poisson
(b) Gamma, $b = n$

In general: $\Phi_{\underline{Y}}(\omega) = \Phi_{\underline{X}}^{N}\left(\frac{\omega}{N}\right)$

(a) For Poisson:

$$\Phi_{\underline{X}}(\omega) = e^{a(e^{j\omega}-1)}$$

Thus: $$\Phi_{\underline{Y}}(\omega) = e^{(aN)[e^{j(\frac{\omega}{N})}-1]} \qquad (1)$$

Since $e^{-b}\sum_{n=0}^{\infty} \frac{b^n}{n!}\delta(x-n)$ Fourier transforms to $\exp[b(e^{j\omega}-1)]$ then, from the scaling theorem, (1) inverse transforms to:

$$f_{\underline{Y}}(x) = N e^{-(aN)} \sum_{n=0}^{\infty} \frac{(aN)^n}{n!}\delta(Nx-n)$$

$$= e^{-aN}\sum_{n=0}^{\infty}\frac{(aN)^n}{n!}\delta\left(x-\frac{n}{N}\right)$$

$$\overline{y} = \int_{-\infty}^{\infty} y\, f_{\underline{Y}}(y)dy$$

$$= e^{-aN}\sum_{n=0}^{\infty}\left(\frac{n}{N}\right)\frac{(aN)^n}{n!}$$

set $m = n-1$

$$\Rightarrow \overline{y} = \frac{1}{N}e^{-aN}\sum_{m=0}^{\infty}\frac{(aN)^{m+1}}{m!}$$

$$= \frac{1}{N}e^{-aN}(aN)e^{aN}$$

$$= a = \overline{x}$$

$$\overline{y^2} = e^{-aN} \sum_{n=0}^{\infty} \left(\frac{n}{N}\right)^2 \frac{(aN)^n}{n!}$$

$$= e^{-aN} \frac{1}{N^2} \sum_{n=1}^{\infty} \frac{n(aN)^n}{(n-1)!}$$

but: $\int^{aN} \sum_{n=1}^{\infty} \frac{n(aN)^{n-1}}{(n-1)!} d(aN) = \sum_{n=1}^{\infty} \frac{(aN)^n}{(n-1)!} = (aN)e^{aN}$

thus: $\sum_{n=1}^{\infty} \frac{n(aN)^{n-1}}{(n-1)!} = (1+aN)e^{aN}$ and

$$\overline{y^2} = e^{-aN} \frac{1}{N^2} \quad aN\,[1+aN]\,e^{aN}$$

$$= \frac{a(1+aN)}{N}$$

Thus:

$$\sigma_Y^2 = \overline{y^2} - \bar{y}^2 = \frac{a(1+aN)}{N} - a^2$$

$$= \frac{a}{N} + a^2 - a^2 = \frac{a}{N} = \frac{\text{var } x}{N}$$

(b) For GAMMA:

$$f_X(x) = \frac{c^{b+1}}{\Gamma(b+1)} x^b e^{-cx} \mu(x)$$

$$= \frac{c^{n+1}}{n!} x^n e^{-cx} \mu(x)$$

from (5.72) on p. 154:

$$\Phi_X(\omega) = \frac{c^{n+1}}{(c-j\omega)^{n+1}}$$

Thus:

$$\Phi_Y(\omega) = \frac{c^{N(n+1)}}{(c-j\frac{\omega}{N})^{N(n+1)}}$$

For $m+1 = N(n+1)$, this inverse transforms to:

$$f_Y(x) = N \frac{c^{m+1}}{m!} (Nx)^m e^{-c(Nx)} \mu(x)$$

or:

$$f_Y(x) = \frac{N c^{N(n+1)}}{[N(n+1)-1]!} (Nx)^{N(n+1)-1} e^{-(cN)x} \mu(x)$$

or:

$$f_{\underline{Y}}(x) = \frac{(NC)^{N(n+1)}}{[N(n+1)-1]!} x^{N(n+1)-1} e^{-(NC)x} \mu(x)$$

This is a gamma density with parameters:

$$\hat{b} = N(n+1)$$

$$\hat{c} = NC$$

The mean is thus: (pg 147)

$$\bar{y} = \frac{\hat{b}+1}{\hat{c}} = \frac{n+1}{c} = \bar{x}$$

$$\text{var}\,\underline{Y} = \frac{\hat{b}+1}{\hat{c}^2} = \frac{n+1}{Nc^2} = \frac{\text{var}\,\underline{X}}{N}$$

Here, as in all previous examples,

$$\bar{y} = \bar{x} \quad \text{and} \quad \text{var}\,\underline{Y} = \frac{1}{N}\text{var}\,\underline{X}$$

Homework:

1. For $\underline{Y} = \frac{1}{N}\sum_{n=1}^{N} \underline{X}_n$, show:

$$\bar{y} = \bar{x} \quad , \quad \text{var}\,\underline{Y} = \frac{1}{N}\text{var}\,\underline{X}$$

assuming everything exists.

2. For the Cauchy distribution, discuss what happens when we attempt to evaluate the mean and variance by differentiating the characteristic function.

Also: Chapt 6: 2, 4, 6, 7, 8

Chapt 7: 1, 2, 3, 4, 5

Solutoms to midterm #2

$Z = (-1)^{X+Y}$

$E[Z] = E[(-1)^{X+Y}] = E[(-1)^{X}]\ E[(-1)^{Y}]$

$E[(-1)^{X}] = \sum_{n=0}^{\infty} (-1)^n \frac{e^{-a} a^n}{n!} = e^{-a} \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} = e^{-2a}$

Therefore: $E[Z] = \exp[-2(a+b)]$

$E Z^2 = E[\{(-1)^{X+Y}\}^2] = 1$

$\therefore$ var $Z = 1 - \exp[-4(a+b)]$

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = 2\int_{y=0}^{\sqrt{a^2-x^2}} dy/\pi a^2 = \frac{2\sqrt{a^2-x^2}}{\pi a^2};\ |x| \le a$

$\therefore f(y/x) = f(x,y)/f_X(x)$

$= \frac{1/\pi a^2}{2\pi a^2\sqrt{a^2-x^2}} = [2(\pi a^2)^2\sqrt{a^2-x^2}]^{-1};\ |r| \le a$

and $E[Y/X] = \int_{-\infty}^{\infty} y\, f(y/x)\,dy = 0$

(this is because $f(y/x)$ is not explicitly a function of y & the integration limits are over $-\sqrt{a^2-x^2} \le y \le \sqrt{a^2+y^2}$)

var $Y/X = E[Y^2/X] = \int_{-\infty}^{\infty} y^2 f(y/x)\,dx$

$= \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2 \frac{dy}{2(\pi a^2)^2\sqrt{a^2-x^2}}$

$= \frac{1}{(\pi a^2)^2\sqrt{a^2-x^2}} \cdot \int_0^{\sqrt{a^2-x^2}} y^2 dy^2$

$= \frac{1}{(\pi a^2)^2\sqrt{a^2-x^2}} \cdot \frac{1}{3}(a^2-x^2)^{3/2}$

$= \frac{a^2-x^2}{3(\pi a^2)^2};\quad r < a$

This is minimum when $X = \pm a$. Then, we <u>must</u> require that $Y = 0$ and

var $Y/X = 0$. This makes sense!

$$g_n = \sum_{k=1}^{N} t_{nk}\, f_k$$

$$E\, g_n = \sum_{k=1}^{N} t_{nk}\, E\, f_k = 0$$

$$E\, g_n^2 = \text{var}\, g_n = E\left[\left(\sum_{k=1}^{N} t_{nk}\, f_k\right)^2\right]$$

$$= E\left[\sum_{k=1}^{N}\sum_{\ell=1}^{N} t_{nk}\, t_{n\ell}\, f_k\, f_\ell\right]$$

$$= \sum_{k=1}^{N}\sum_{\ell=1}^{N} t_{nk}\, t_{n\ell}\, \sigma^2 \delta_{k-n} = \sigma^2 \sum_{k=1}^{N} (t_{nk})^2$$

$$= 1$$

$Pr\,[|g_n| \le 1] \simeq 2\,\text{erf}(1)$ by CLT

$$z(t) = e^{jXt}$$

$$E\, z(t) = E\, e^{jXt} = e^{-\sigma^2 t^2/2}\, e^{j\eta t} = \eta_z(t)$$

(follows from def of characteristic function for a normal r.v.).

$$E\, z(t_1)\, z^*(t_2) = R(t_1, t_2)$$

$$= E\, e^{jX(t_1 - t_2)}$$

$$= e^{-\sigma^2 (t_1 - t_2)^2/2}\, e^{j\eta(t_1 - t_2)}$$

or

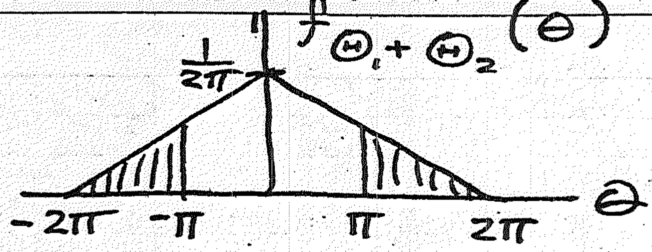
$$R(\tau) = e^{-\sigma^2 \tau^2/2}\, e^{j\eta\tau}$$

Then

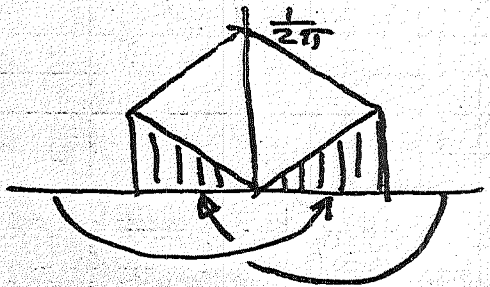
$$\text{var}\, z(t) = R(0) = 1$$

The process is <u>not</u> stationary since

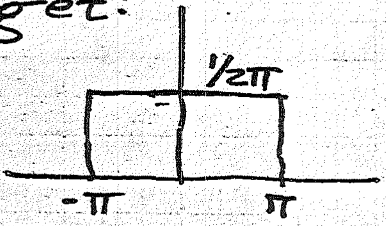
$$\eta_z(t) \ne \text{constant}$$



modulo $2\pi \rightarrow$ move masses

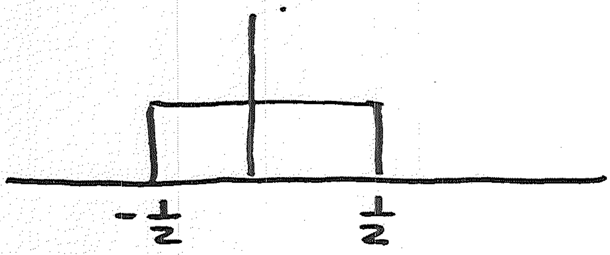


add to get:



∴ Uniform on $-\pi$ to π.

How can we generate



A: $Y = X - \frac{1}{2}$

What about a dice roll?

$$D_n = \text{Int}\left[6X_n + 1\right]$$

~~How about the sum of two dice?~~

~~$D_n + D_{n+1}$ = sum~~

Gaussian R.V.

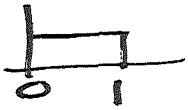
(a) can find the $g \ni Y_n = g(X_n)$
is gaussian (ugly!)

(b) Central limit theorem

$$Y = X_1 + X_2 + X_3 + \ldots + X_N$$

$$\simeq \text{Mean} = \frac{N}{2}, \quad \text{var} = \frac{N}{12}$$

(c) Compute X_n & X_{n+1}



then

$$Y_n = (-2 \ln X_1)^{\frac{1}{2}} \cos 2\pi X_1$$

$$Y_{n+1} = (-2 \ln X_1)^{\frac{1}{2}} \sin 2\pi X_2$$

Y_n & Y_{n+1} are zero mean -unit variance normal r.v.'s.

Discrete Methods

Generating Random Numbers

Use Table

Pseudo-Random numbers

Congruence Method of generating ~~pse~~ pseudo-random numbers

$$X_{n+1} = (a X_n + b)_{\text{MOD } T}$$

b & T should be relatively prime

Example: $a = \frac{3}{2}$, $b = \pi$, $T = 1$

(R) Seed $X_0 = 1$ ⟸ NOTE: CAN GET FROM RANDOM # TABLE

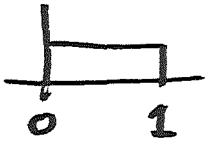
$X_1 = 0.1415926$

$X_1 = 0.641592654$

$X_2 = 0.103981635$

$X_3 = 0.297565$

Uniform



Can Show:

$$\rho_s = E[X_n X_{n+s}]$$

$$= \frac{1 - 6 \frac{b_s}{T}\left(1 - \frac{b_s}{T}\right)}{a_s} + \epsilon$$

$$a_s = a^s \quad (\text{mod } T)$$

$$b_s = \left(\sum_{n=0}^{s-1} a^n\right) b \qquad (\text{mod } T)$$

$$|\epsilon| < a_s / T$$

HP $\quad \underline{X}_{n+1} = \text{Frc}(\underline{X}_n + \pi)^5$

MEWORK #1

CHAPT 2: 3, [illegible], 8, $\boxed{15}$

CHAPT 3: 2, 3, ④

MEWORK #2 3: 7, 9

HAPT 4: 2, 6, 7, 8, [illegible]

MEWORK #3

CHAPT 5: 1, 2, 4, 7, ~~8~~, 10

MEWORK #4

CHAPT. 5: (4, 7, 8, 10, 16, 17, 18

MEWORK #5

CHAPT. 6: 2, [illegible], 7, 8

CHAPT. 7: 1, 2, 3, 4, 5

MEWORK #6

HAPT. 7: 7, [illegible], ~~15~~, 16, 22, 26

HAPT. 8: 4, [illegible], 8, 13, 19, 20

~~HAPT 9: 1, 2, 7, 10~~

. #7. Chapt 9: 1, 2, 4ab, [illegible] 10

MEWORK #7 Chapt 9: 12, 13, 16, 17, 22a

HAPT. 10: 1, 5, 8, 10, 17

Find the distribution, mean and variance of

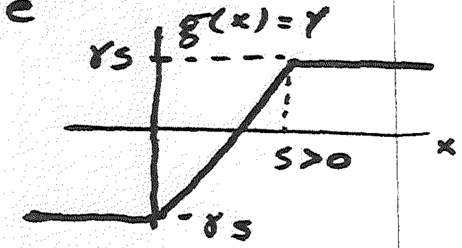
$$Y = \frac{1}{N} \sum_{n=1}^{N} X_n$$

when X_n is

(a) Poisson

(b) Gamma

$$X \sim \frac{\alpha}{2} e^{-\gamma|x|}$$



$$f_Y(x) = \frac{1}{2}\,\delta(x+\gamma s) + \frac{1}{4} e^{-\frac{1}{2}(\gamma s + x)}\,\mathrm{rect}\left(\frac{x}{2\gamma s}\right) + \frac{1}{2} e^{-\gamma s}\,\delta(x-\gamma s)$$

GIVEN A RANDOM TELEGRAPH SIGNAL FROM A POISSON PROCESS, FORM A NEW RANDOM PROCESS

$$Y(t) = (-1)^N X(t)$$

where $Pr[N=0] = Pr[N=1] = \frac{1}{2}$

find the mean of $Y(t)$

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	Thomas	"An Intro to Statistical Communication Theory" (Wiley, 1969)
	Larson	"Intro. to Probability Theory and Statistical Inference" (Wiley)
	Bracewell	"The Fourier Transform and Its Applications 2nd Ed" (McGraw-Hill, 1978)
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& STOCHASTIC PROCESSES

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1. Classic Def

$P(a) = n_a / n$

~~REVIEW OF PROBABILITY~~

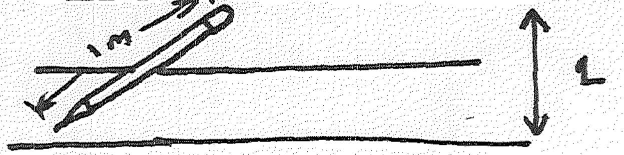
~~Pr[event]~~

~~Definitions:~~

1 ~~a~~. Relative Frequency (EXPERIMENT ~~[illegible]~~)

$$P(a) = \lim_{n \to \infty} \frac{n_a}{n}$$

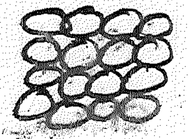
Ex a = pencil crosses line



Ex: Roll 20 dice

a: Event that sum = 30

Ex



$Pr[\text{IN CIRCLE}] = \frac{\pi}{4} = \frac{\#\text{IN}}{\#\text{TOTAL}}$

~~2. Re~~ Ex

Monte-Carlo Sim

M.S.E. (Mean Square estimation) of a r.v. by a const.

$$\hat{Y} = a$$

MINIMIZE ESTIMATE'S UNCERTAINTY:

$$E\left[(Y - a)^2\right] = \int_{-\infty}^{\infty} (y-a)^2 f_Y(y)\,dy$$

$= \cancel{\overline{Y^2} - 2a\overline{Y} + \overline{Y}^2}$ $\quad \overline{Y^2} - 2a\overline{Y} + a^2$

can't change moments. Let

$$a = \overline{Y} = \text{mean}$$

$$\frac{\delta}{\delta a} = -2\overline{Y} + 2a = 0 \Rightarrow a = \overline{Y}$$

Nonlinear:

Mean-square estimation

~~w.~~ From a r.v. X, we wish to estimate the r.v. Y, by

$$\hat{Y} = g(X).$$

Question: How to choose g?

Answer: Minimize m.s.e (mean square estimation):

$$E\left[\left(Y - g(X)\right)^2\right]$$

$$= \iint_{-\infty}^{\infty} \left(y - g(x)\right)^2 f_{XY}(x,y)\,dx\,dy$$

Note: MINIMIZING SQUARE OF "DISTANCE" 'TWIXT THEM.

Note: $F_{XY}(x,y) = F_X(y|x)\,F_X(x)$

$\Rightarrow \quad f_{XY}(x,y) = f_Y(y|x)\,f_X(x)$

Thus:

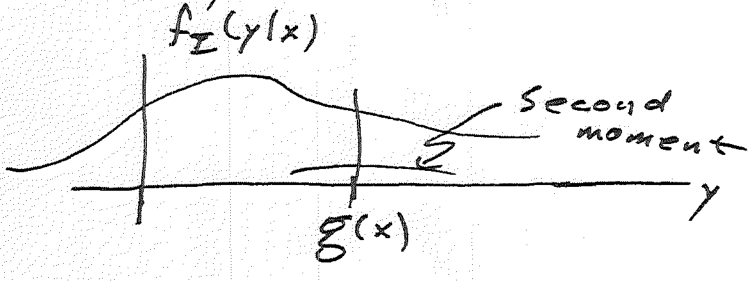
$$E\left[\left(Y - g(X)\right)^2\right] = \int_{-\infty}^{\infty} f_X(x) \int_y \left(y - g(x)\right)^2 f_Y(y|x)\,dx\,dy$$

Minimize$_g$ $\forall x$

$$\int_y \left(y - g(x)\right)^2 f_Y(y|x)\,\cancel{dx}\,dy \leftarrow$$

Second moment of $f_Y(y|x)$ about the point $y = g(x) = \text{const}$

$$= \int_y y^2 f_Y\left(y + g(x) \,|\, x\right) dy$$



Recall:

$$\int_{-\infty}^{\infty} (y-a)^2 f_{\mathbf{Y}}(y)\,dy$$

= second moment of y about $y=a$

$a = E(y)$ minimizes

Here:

$g(x) = E[Y/x]$ is mini·

Thus, best m.s. es. is:

$g(x) = E[Y/X]$

$E[\mathbf{Y}/\mathbf{X}]$ = regression curve

If $\mathbf{X}$ & $\mathbf{Y}$ are ind:

$E[\mathbf{Y}/\mathbf{X}] = E[\mathbf{Y}] \Rightarrow$ no help if x known

Ex

$$f_{XY}(x,y) = e^{-x}\, x\, e^{-xy} \mu(x) \mu(y)$$

$$f_{XY}(x,y) = f_Y(y/x)\, f_X(x)$$

$$f_X(x) = x \int_{y=0}^{\infty} e^{-xy} \mu(y)\, dy\, \mu(x)\, e^{-x}$$

$$= [illegible]$$

$$= e^{-x} \mu(x)$$

$$\Rightarrow f_Y(y/x) = x e^{-xy} \mu(x) \mu(y)$$

$$E[Y/X] = x \int_0^{\infty} y\, e^{-xy} \cancel{\mu(y)}\, dy\, \mu(x)$$

$$\int^x \frac{E[Y/X]}{x} = -\int_0^{\infty} e^{-xy}\, dy$$

$$= -\frac{1}{x}$$

$$E[Y/X] = \cancel{X} \ominus \times \frac{1}{x^2} = \frac{1}{x}$$

Linear Mean-Square Estimation

assume $g(x) = ax + b$

Not as good $\leftrightarrow$ generally more tractible

$\Rightarrow$ Minimize (a & b) $E\left[\left(Y - (aX + b)\right)^2\right] = e$

$$e = \iint (y - ax - b)^2 f_{XY}(x,y)\,dx\,dy$$

optimum:

$$a = \frac{r\sigma_Y}{\sigma_X} \quad ; \; b = \overline{y} - a\overline{x} \quad ; \; r = \frac{\overline{xy}}{\sigma_x \sigma_y}$$

= correlation coefficient

gives

$$e_m = \sigma_Y^2 (1 - r^2)$$

Proof: for given a, choose

$$b = \overline{y} - a\overline{x}$$

$b = \overline{y} - a\overline{x}$ ← obvious choice

Then:

$$E[(Y - aX - b)^2]$$
$$= E\left[\left((Y - \overline{y}) - a(X - \overline{x})\right)^2\right]$$
$$= \sigma_Y^2 - 2r\sigma_X \sigma_Y a + \sigma_x^2 a^2$$

Take $\frac{d}{da} = 0$

gives

$$a = r\sigma_Y / \sigma_x$$

Note: If X & Y are zero mean, prob. becomes:

Minimize

$$E\left[\{Y - aX\}^2\right]$$

gives $a = \frac{r\sigma_Y}{\sigma_X}$; $e_m = \sigma_Y^2 (1 - r^2)$

for X & Y zero mean

Proof: From previous result:

$E[(Y - aX)^2]$ is

min for $a = \frac{r\sigma_y}{\sigma_x}$

Thus

$$E\left[\left(Y - \frac{r\sigma_y}{\sigma_y}X\right)X\right] \overset{?}{=} 0$$

$$= \sigma_x\sigma_y r - \frac{r\sigma_y}{\sigma_x}\sigma_x^2$$

Q.E.D.

Note:

$$E\left[\left(Y - X\frac{r\sigma_y}{\sigma_x}\right)Y\right)$$

$$= \sigma_y^2 - r\sigma_x\sigma_y\frac{r\sigma_y}{\sigma_x}$$

$$= \sigma_y^2 - r^2\sigma_y^2 = (1-r^2)\sigma_y^2$$

~~Alternate proof~~:

~~Note~~

~~$\epsilon_m = E[Y^2] - \left(\frac{r\sigma_y}{\sigma_x}\right)^2 E[X^2]$~~

Two R.V. are "orthogonal" if

$$\overline{xy} = E[\underline{X}\underline{Y}] = 0$$

Suff cond: x & y are ind and one is zero mean

ORTHOGONALITY PRINCIPLE: $\underline{X}$ & $\underline{Y}$ zero mean The const a that minimizes:

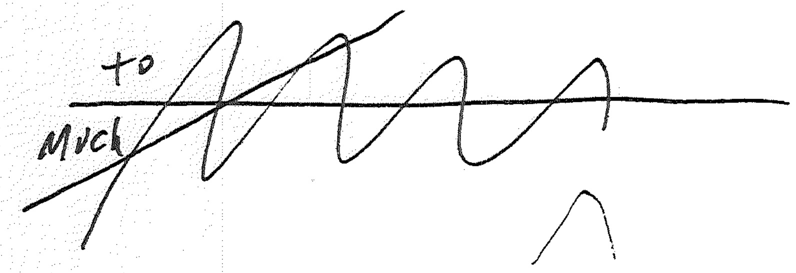
$$e = E\left[(\underline{Y} - a\underline{X})^2\right]$$

is such that $\underline{Y} - a\underline{X}$ is orthogonal to $\underline{X}$. ie

$$E\left[(\underline{Y} - a\underline{X})\,\underline{X}\right] = 0$$

Then: $\epsilon_m = E\left[(\underline{Y} - a\underline{X})\,\underline{Y}\right]$

~~Hilbert space inter:~~



8. Sequences of Random Variables

$$F(x_1, x_2, \ldots x_n) = P_r[X_1 \le x_1, \ldots X_n \le x_n]$$

$$f(x_1 \ldots x_n) = \frac{\delta^n}{\delta x_1 \cdots \delta x_n} F(x_1 \ldots x_n)$$

$$f(x_1) = \int_{x_2} \int_{x_3} \cdots \int_{x_n} f(x_1 \ldots x_n)\, dx_1 \ldots dx_n$$

~~INDEP IF~~

~~$F(x_1 \ldots x_n) =$~~ ~~$f_1(x_1) \ldots f_n(x_n)$~~

~~$f(x_1 \ldots x_n) =$~~

$(x_1, \ldots x_n)$

Uncorrelated: $E[x_i x_j] = E[x_i]\, E[x_j]\ ;\ i \neq j$

Orthogonal: $E[x_i x_j] = \delta_{ij}\, \overline{x_i^2}$

Uncor. & Orth. r.v.'s are invariant under linear combination. ie

$a_1 x_1 + \ldots + a_K x_K$ is uncor (orth)

to $a_{K+1} x_{K+1} + \cdots + a_n x_n$

(Elaborate)

For uncorrel:

$\sigma^2_{x_1 + \cdots + x_n}$

~~$= E\left[\left((X_1 - \bar{x}_1) + X_2 + X_3 + \ldots + X_n\right)^2 - E(\bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_n)\right])^2\Big]$~~

~~$= E\left[\left\{(X_1 + \ldots + X_n) - (\bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_n)\right\}^2\right]$~~

$= E\left[\left(\sum_{n=1}^{N} (X_n - \bar{x}_n)\right)^2\right]$

$= E\left[\left(\sum_{n=1}^{N} X_n - \sum_{n=1}^{N} \bar{x}_n\right)^2\right]$

$= E\left[\left(\sum_{n=1}^{N} X_n\right)^2 - 2 \sum_{n=1}^{N} \bar{x}_n \sum_{n=1}^{N} X_n + \left(\sum_{n=1}^{N} \bar{x}_n\right)^2\right]$

$= E\left[\left(\sum_{n=1}^{N} X_n\right)^2 - \left(\sum_{n=1}^{N} \bar{x}_n\right)^2\right]$

~~$= E\left[\sum_n \sum_m X_n X_m - \sum_n \overline{x}_n^2\right]$~~

~~$= E\left[\sum_n X_n^2\right.$~~

But $E\left[\left(\sum_n X_n\right)^2\right] = E\left[\sum_n \sum_m X_n X_m\right]$

$= E\left[\sum_n X_n^2\right] - \sum_n \sum_{m \neq n} \overline{x_n}\ \overline{x_m}$

$\Rightarrow \sigma_{x_1 + \ldots x_n} = E\left[\sum_n X_n^2 - \sum_n \sum_{m \neq n} \overline{x_n x_m}\right.$

$\left. - \left(\sum \overline{x_n}^2 - \sum_n \sum_{m \neq n} \overline{x_n}\, \overline{x_m}\right)\right]$

$= \sigma^2_{x_1} + \sigma^2_{x_2} + \ldots$

IF orthogonal:

$$E\left[\left(\sum_{n=1}^{N} x_n\right)^2\right] = \sum E\left[x_n^2\right]$$

MEAN SQUARE ESTIMATION.

$$x_0, x_1, \dots, x_n$$

wish to estimate x_0 in terms of $x_1, \dots x_n$

$\hat{x}_0 = g(x_1 \dots x_n)$

MINIMIZE M.S. error:

$$E\left[\left(x_0 - g(x_1 \dots x_n)\right)^2\right]$$

Generalization of previous result:

Choose:

$$g(x_1 \dots x_n) = E\left[x_0 \mid x_1 \dots x_n\right]$$

MUST know density of

$x_0 \dots x_n$

Φ_X defined

SAMPLE MEANS & ESTIMATES FROM CHAR. FUNC.

$$\overline{x} = \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$$

$$\overline{V} \overset{\equiv}{\cancel{\text{var}\, x}} = \frac{1}{n}\left[(x_1 - \overline{x})^2 + \ldots + (x_n - \overline{x})^2\right]$$

$x_i \sim f_X(x)$

x's ARE INDEPENDENT.

(EXPLAIN SAMPLES)

ASSUME:

$$\left.\begin{array}{l} E[x_i] = \eta \\ \text{var}\ x_i = \sigma^2 \end{array}\right\} \forall i = 1, 2, \ldots, n$$

ALTERNATE WILL SHOW: $E[\overline{x}] = \eta \qquad \sigma_{\overline{x}}^2 = \frac{\sigma^2}{n}$

$\cancel{E[\overline{V}]}$

$$E[\overline{V}] = \frac{n-1}{n}\sigma^2$$

PROOF

$$E(\bar{x}) = \frac{1}{n} E\left[\sum^{n} x_i\right]$$
$$= \frac{1}{n} \sum E[x_i]$$
$$= \frac{1}{n}\, n\, \eta = \eta$$

~~$\sigma_{\bar{x}}^2 = E[$~~

OR: $E\left[e^{j\omega\bar{x}}\right] = \Phi_{\bar{x}}(\omega)$

$$= E\left[e^{j\omega\frac{1}{n}\sum x_i}\right]$$
$$= E\left[\prod_{i=1}^{n} e^{j\omega x_i/n}\right]$$
$$= \prod_{i=1}^{n} E\left[e^{j\omega x_i/n}\right]$$
$$= \Phi_x^n\left(\frac{\omega}{n}\right)$$

$$\Rightarrow \Phi_{\bar{x}}(\omega) = \Phi_x^n\left(\frac{\omega}{n}\right)$$

$$\frac{d\Phi_{\bar{x}}}{d\omega} = n \cdot \frac{1}{n} \Phi_x^{n-1}\left(\frac{\omega}{n}\right) \Phi_x'\left(\frac{\omega}{n}\right)$$

$$@\omega = 0 \Rightarrow \eta_{\bar{x}} = 1 \times \eta_x$$

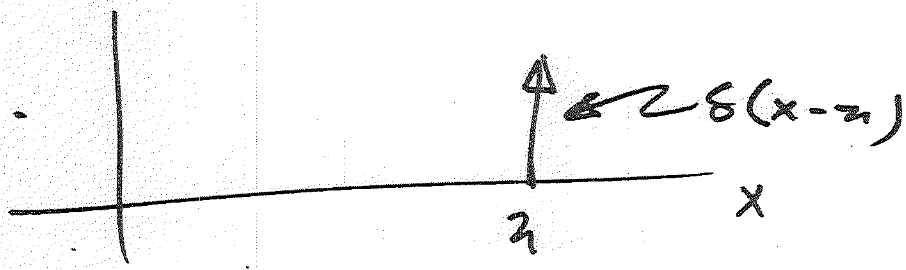
$$\frac{d^2\phi_{\bar{x}}}{d\omega} = \frac{n-1}{n} \Phi_x^{n-2}\left(\frac{\omega}{n}\right)\left[\Phi'\left(\frac{\omega}{n}\right)\right]^2 + \frac{1}{n} \Phi_x^{n-1}\left(\frac{\omega}{n}\right) \phi''\left(\frac{\omega}{n}\right)$$

$$@\omega = 0 \Rightarrow E(\bar{x}^2) = \frac{n-1}{n}\eta^2 + \frac{1}{n}(\sigma^2 - \eta^2)$$

$$\sigma_{\bar{x}}^2 = \left(1 - \frac{1}{n}\right)\eta^2 + \frac{1}{n}(\sigma^2 + \eta^2) - \eta^2$$
$$= -\frac{1}{n}\eta^2 + \frac{1}{n}\sigma^2 - \frac{\eta^2}{n}$$
$$= \frac{\sigma^2}{n}$$

$\lim_{n\to\infty} \frac{\sigma^2}{n} = 0 \Leftarrow$ variance of δ

IN LIMIT:



MARKOFF'S PROOF OF "LAW OF LARGE #S"

(p. 265)

Elaborate

~~Central Limit Theorem~~

~~$\sigma_1^2 + \sigma_2 \ldots + \sigma_n^2 \to \infty$~~

~~$\exists\ \alpha > 0$~~

~~$\ni \int_{-\infty}^{\infty} x^\alpha f_i(x)\,dx < C$~~

$\text{var}\, X_i = \sigma^2$

Sample mean & variance ; $x_i \sim f_X(x)$, $\overline{x_i} = \eta$

$$\bar{x} = \frac{1}{N} \sum^{N} \bar{x}_n$$

$$\bar{V} = \frac{(x_1 - \eta) + \ldots + (x_n - \eta)^2}{N}$$

Ind. → uncorr:

$$E[(X_i - \bar{x})(X_k - \bar{x})] = \delta_{ik}$$

TO PROOVE: ① $E\bar{x} = \eta$, ② $\text{var}\,\bar{x} = \sigma^2/N$

③ $E\,\bar{V} = \frac{n-1}{n}\sigma^2$

PROOF:

① $E[\bar{x}] = \frac{E[\sum \bar{x}_n]}{n} = \eta$

since uncorrelated:

② $\sigma_{\bar{x}}^2 = \frac{1}{N^2}(\sigma_x^2 + \ldots \sigma_{x_n}^2)$

$= \frac{\sigma^2}{N}$

③ Lastly:

Lemma $E[(x_i - \eta)(\bar{x} - \eta)] = E\left[(x_i - \eta)\, \frac{(x_i - \eta) + \ldots (x_n - \eta)}{N}\right]$

$= E\left[(x_i - \eta)\, \frac{(x_i - \eta)^*}{N}\right]$ ← since ~~uncor~~ orth (unc & zero mean)

$= \frac{\sigma^2}{N}$

Thus:

$$E[(x_i - \bar{x})^2] = E\left[\left((x_i - \eta) - (\bar{x} - \eta)\right)^2\right]$$

$= \sigma^2 - 2\frac{\sigma^2}{N} + \frac{\sigma^2}{N}$

$= \frac{N-1}{N}\sigma^2$

Thus:

$$E[\bar{V}] = E\left[\frac{(X_1 - \bar{x})^2 + \ldots + (X_n - \bar{x})^2}{N}\right]$$

$= \frac{(N-1)\sigma^2}{N} \cdot \frac{N}{N}$ Q.E.D.

NORMAL RANDOM VARIABLES

N^{th} order zero mean; equal variance σ^2

$$f_{x_1,\dots x_n}(x_1 \dots x_n) = \frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n} e^{-\frac{x_1^2+x_2^2+\dots+x_n^2}{2\sigma^2}}$$

Let:

$$\bar{x} = \frac{x_1+\dots+x_n}{n}$$

$$\bar{v} = (x_1-\bar{x})^2+\dots+(x_n-\bar{x})^2 / n$$

$$\chi = \sqrt{x_1^2+\dots+x_n^2}$$

$$y = \chi^2 = x_1^2+\dots+x_n^2$$

Densities for each:

$$f_{\bar{x}} \sim \text{normal}\left(0, \text{var} = \frac{\sigma^2}{N}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2/N}} e^{-nx^2/2\sigma^2}$$

For χ

$$\chi \le \sqrt{x_1^2+\dots x_n^2} < \chi + d\chi$$

Hypershell

$dv \propto \chi^{n-1} d\chi$

ex circle
$n=2$
$2\pi r$
sphere
$4\pi r^2$

In shell; mass is:

$$\frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n} e^{-\chi^2/2\sigma^2}$$

$$\Rightarrow f_\chi(\chi)\,d\chi = P[\chi \le \sqrt{x_1^2+\dots+x_n^2} < \chi+d\chi] = \frac{\text{CONST}}{(2\pi)^{\frac{n}{2}}\sigma^n} \chi^{n-1} e^{-\chi^2/2\sigma^2} d\chi$$

can find CONST via unit area:

$$f_\chi(\chi) = \frac{2}{2^{n/2}\sigma^n \Gamma\left(\frac{n}{2}\right)} \chi^{n-1} e^{-\chi^2/2\sigma^2} \mu(\chi)$$

For $y = x^2$:

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + \cancelto{0}{f_X(-\sqrt{y})} \right]$$

$$f_Y(y) = \frac{1}{2^{\frac{n}{2}}\sigma^n \Gamma(\frac{n}{2})} \cancel{x^{n-1}\, e^{-}}$$

$$y^{\frac{n-2}{2}} e^{-y/2\sigma^2} \mu(y)$$

(Special case of Gamma) $\uparrow$ χ^2 density

specified by: σ^2

n = degrees of freedom

For $n=2$

$$\overline{V} = \frac{1}{2}\left(x_1 - \frac{x_1+x_2}{2}\right)^2 + \frac{1}{2}\left(x_2 - \frac{x_1+x_2}{2}\right)^2$$

$$= \left(\frac{x_1-x_2}{2}\right)^2$$

but $z_1 = \frac{x_1-x_2}{2} \longrightarrow$ zero mean

$$\Rightarrow \overline{V}_B \sim \chi_1^2$$

~~For n=3~~

Papoulis has $n=3$

OMIT

Alternate treatment: $X_n \sim \mu$ mean, σ^2 normal

$$\overline{v} = \frac{1}{N}\sum_{n=1}^{N}\left(X_n - \overline{X}\right)^2$$

$$\overline{x} = \frac{1}{N}\sum_{n=1}^{N}\left(X_n\right)$$

Note:
if $Y \sim \chi^2_{n-1}(\sigma^2)$
$\Rightarrow \frac{Y}{\sigma^2} \sim \chi^2_{n-1}(1)$

Showed in H.W:

$\overline{v}$ & $\overline{x}$ are ind.

Proof that $\frac{Ns^2}{\sigma^2}$ is χ^2_{n-1}

We know:

$$\frac{1}{N}\sum_{n=1}^{N}\left(X_n - \eta\right)^2 \sim \chi^2_n \qquad (\sigma^2 = 1)$$

Now

$$\sum_{n=1}^{N}\left(X_i - \mu\right)^2 = \sum_{n=i}^{N}\left(X_i - \overline{x} + \overline{x} - \mu\right)^2$$

$$= \sum_{n=1}^{N}\left(X_i - \overline{x}\right)^2 + N\left(\overline{x} - \mu\right)^2$$

(since $2(\overline{x} - \mu)\sum_{n=1}^{N}\left(X_i - \overline{X}\right) = 0$

Thus;

$$\frac{\sum_{n=1}^{N}\left(X_i - \mu\right)^2}{\sigma^2} = \cancel{\frac{\sum_{n=1}^{N}\left(X_i - \overline{x}\right)^2}{\sigma^2} +}$$

$$= \frac{N(\overline{x} - \mu)^2}{\sigma^2} + \frac{N\overline{V}}{\sigma^2}$$

$$\frac{\sum_{n=1}^{N} (X_i - \mu)^2}{\sigma^2} \longrightarrow \chi^2_n$$

$$\left(\frac{\bar{x} - \mu}{\sigma/\sqrt{N}}\right)^2 \longrightarrow \chi^2_1$$

Also, ind (showed in class)

Them:

$$\left.\begin{array}{l} Y \sim \chi^2_n \\ Z \sim \chi^2_m \end{array}\right\} \sigma^2 = 1$$

$$Y + Z = \chi^2_{n+m}$$

(Proove via char. functions)

Hence $\Longrightarrow \dfrac{N\bar{V}}{\sigma^2} \sim \chi^2_{n-1}$

Not zero mean

Recall

$$\overline{V} = \frac{(x_1-\bar{x})^2 + \ldots + (x_n - \bar{x})^2}{N}$$

$$= z_1^2 + z_2^2 + \ldots + z_{n-1}^2$$

$z =$ ~~$\frac{x_i - \bar{x}}{\sqrt{n}}$~~ zero mean, ~~unit variance~~ gaussian r.v.

Why $n-1$, $z_n \rightarrow$ not n z_n's (since there are n x_n's)

$$\overline{V} = c_1^2 + \ldots + c_n^2 \qquad c_n = \frac{x_i - \bar{x}}{\sqrt{n}}$$

But c_i's not ind.

$$\sum_{i=1}^{n} c_i = 0$$

~~let~~ Since

$$c_n = -(c_1 + \ldots + c_{n-1})$$

substituting & Complete square results in $\overline{V}$ expression } Papoulis Gives ex for $n=3$

~~z's a~~

z's zero mean, normal with zero mean & equal variance

$$\Longrightarrow \overline{V} \sim \chi^2_{n-1} \leftarrow \text{special case of gamma}$$

Turns out:

~~But~~ $E(\overline{V}) = \frac{n-1}{n}\sigma^2$

~~$E(\bar{x})$ $E(\bar{x}^2) = \frac{\sigma^2}{n}$ ← Var in $\bar{x}$~~

$$f_{\overline{V}}(v) = \frac{N^{\frac{n-3}{2}}}{2^{(n-1)/2}\left(\frac{\sigma}{\sqrt{n}}\right)^{n-1}\Gamma\left(\frac{n-1}{2}\right)} e^{-\frac{nv}{2\sigma^2}} \mu(v)$$

LINEAR MEAN SQUARE ESTIMATION. MINIMIZE

$$e = E\left[\left(x_0 - (a_1x_1 + a_2x_2 + \dots + a_nx_n)\right)^2\right] \text{ wrt } a_1 \dots a_n$$

Define:

$$R_{ij} = \overline{x_i x_j} = E[x_i x_j]$$

if $\overline{x_i} = 0$, R_{ij} = covariance of x_i & x_j

Ans:

Wish to have error:

$$x_0 - (a_1x_1 + \dots + a_nx_n)$$

is orthogonal to $(x_1 \dots x_n)$. That is:

$$* \; E\left[\{x_0 - (a_1x_1 + \dots + a_nx_n)\} x_i\right] = 0 \; ; i = 1, 2, \dots n$$

Proof:

To minimize, set

$$\frac{\delta e}{\delta a_i} = 0 = \frac{\delta E\left[\left(x_0 - (a_1x_1 + \dots a_nx_n)\right)^2\right]}{\delta a_i}$$

$$= -2E\left(\left[x_0 - (a_1x_1 + \dots + a_nx_n)\right] x_i\right) = 0$$

orthogonality principle follows.

What is min. error?

EXPANDING ~~SQUARE~~ SQUARE & USING ORTH. PRINCIPLE *

~~$e_m = E\left[\left(x_0 - (a_1x_1 + \dots + a_nx_n)\right) x_0\right]$~~

$$e = E\left[\left(x_0 - (a_1x_1 + \dots a_nx_n)\right) \times \left(x_0 - (a_1x_1 + \dots + a_nx_n)\right)\right]$$

$$= E\left[x_0^2 - (a_1x_1x_0 + \dots + a_nx_nx_0)\right]$$

$$= R_{00} - (a_1R_{01} + \dots + a_nR_{0n})$$

Solving for a_i's: from *

$$R_{01} = a_1R_{11} + a_2R_{12} + \dots + R_{1n}a_n$$
$$R_{21} = a_1R_{21} + a_2R_{22} + \dots + R_{2n}a_n$$
$$\vdots$$
$$R_{n1} = a_1R_{n1} + a_2R_{n2} + \dots + R_{nn}a_n$$

OR

$$\vec{R}_0 = [illegible] = \underline{R}\,\vec{a}$$

Central Limit Theorem

Let $X_n \ , n = 1, 2, 3, \ldots, N$

$$E[X_n] = \overline{x_n}$$

$$\text{var } X_n = \sigma_n^2$$

Let:

$$Y_N = \frac{\sum_{n=1}^{N} X_n - \sum_{n=1}^{N} \overline{x_n}}{\sqrt{\sum_{n=1}^{N} \cancel{\text{var } X}\ \sigma_n^2}}$$

p. 214

~~Then~~ Note: $E[Y_N] = 0$

$\text{var } Y_N = 1$ (prove)

Then, as $N \to \infty$, under rather weak conditions:

$$Y_N \sim \text{normal}(0, 1)$$

~~Under~~

Sufficient: $\exists \ \overset{\circ}{m} \ni \overset{\circ}{N}$

$$\infty > \sigma_n^2 > m > 0$$

$$E[|X_n - \overline{X_n}|^3] < M$$

Assures 1 term don't dominate

Special Case:

Identically distributed & ind

$$Y_N = \frac{1}{\sqrt{N}\,\sigma} \sum_{n=1}^{N} \left(X_n - \bar{x} \right)$$

Let $$Z_n = \frac{X_n - \bar{x}}{\sigma}$$

Then: $E[Z_n] = 0$, $\operatorname{var} Z_n = 1$

~~Proof: var $Z_n = E(X$~~

and:

$$Y_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} Z_n$$

$$\Phi_{Y_N}(\omega) = E\left[e^{j\omega Y_N}\right]$$

$$= E\left[e^{j\frac{\omega}{\sqrt{N}}\sum_{n=1}^{N} Z_n}\right]$$

$$= \Phi_Z^N\left[\frac{\omega}{\sqrt{N}}\right]$$

$$\Phi_Z(\omega) = \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!}\,\overline{Z^n}$$

$$= 1 + j\omega\frac{\overline{Z}}{1!} + (j\omega)^2\frac{\overline{Z^2}}{2!} + \frac{\cancel{H.O.T.}\;(j\omega)^3\overline{Z^3}}{3!}$$

$\overline{Z} = 0$, $\overline{Z^2} = \operatorname{var} Z = 1$

$$= 1 + 0 + (j\omega)^2 \frac{1}{2!}\overline{Z^2} + \cancel{H.O.T.}\;\frac{(j\omega)^3\overline{Z^3}}{3!} + H.O.T.$$

$$\Phi_Z\left(\frac{\omega}{\sqrt{N}}\right) = 1 + \frac{(j\omega)^2}{N}\frac{\cancel{\overline{Z^2}}\,1}{2!} + \cancel{H.O.T.}\;\frac{(j\omega)^3\overline{Z^3}}{N^{3/2}\,3!} + \dots$$

Large N:

$$\Phi_Z\left(\frac{\omega}{\sqrt{N}}\right) = 1 - \frac{\omega^2}{2N} \simeq e^{-\frac{\omega^2}{2N}}$$

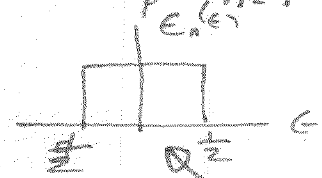
$$\Rightarrow \Phi_{Y_N}(\omega) \longrightarrow e^{-\omega^2/2} \longleftarrow$$ char func of gaussian r.v.

A CLT Example from IRS

~~120~~ 48 number added rounded to nearest dollas

$S = \sum_{n=1}^{N} a_n$ $\quad S' = \sum_{n=1}^{N} \langle a_n \rangle$ ← Rounded

$E = S - S' = \sum_{n=1}^{N} \epsilon_n$



$\text{var}\,\epsilon_n = \frac{1}{12}$

By CLT $E \sim N(0, \frac{N}{12})$

For $N = \cancel{120}\,48$, $E \sim N(0, \cancel{10}\,4)$

$$\Pr[-D \le E \le D] = \Pr[\text{off by no more than \$ } D]$$

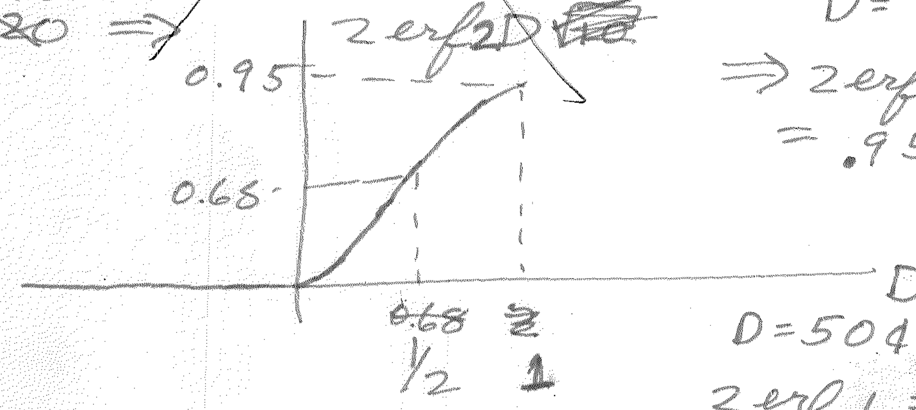
$$= \int_{-D}^{D} \frac{1}{\sqrt{2\pi}\sqrt{N/12}} e^{-\frac{(x)^2}{2(N/12)}} dx$$

$y = \frac{x}{\sqrt{N/12}}$ $\quad d = D/\sqrt{N/12}$

$$= \int_{-D\sqrt{\frac{N}{12}}}^{D\sqrt{N/12}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 2\,\text{erf}\ D\sqrt{\frac{N}{12}}$$

$N = \cancel{120}\,48 \Rightarrow 2\,\text{erf}\,2D$



$D = \$1$

$\Rightarrow 2\,\text{erf}\,2$ (p. 49)

$= .95442$

$D = 50¢$

$2\,\text{erf}\,1 = 0.68$

JOINT CHAR FUNCTION:

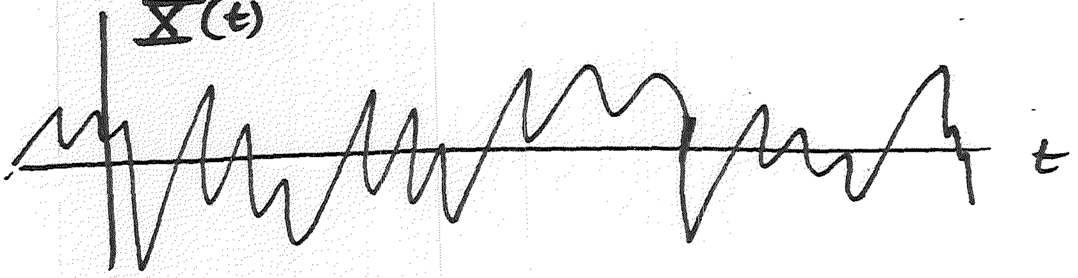
$$\Phi_{XY}(\omega_1, \omega_2) = E\left[e^{j(\omega_1 X + \omega_2 Y)}\right]$$

$$\frac{\delta^k \delta^r \Phi(0,0)}{\delta\omega_1^{\,k}\, \delta\omega_2^{\,r}} = (j)^{k+r}\, m_{kr}$$

(Prove)

If Ind

STOCHASTIC PROCESSES



Random variable depends on time t.

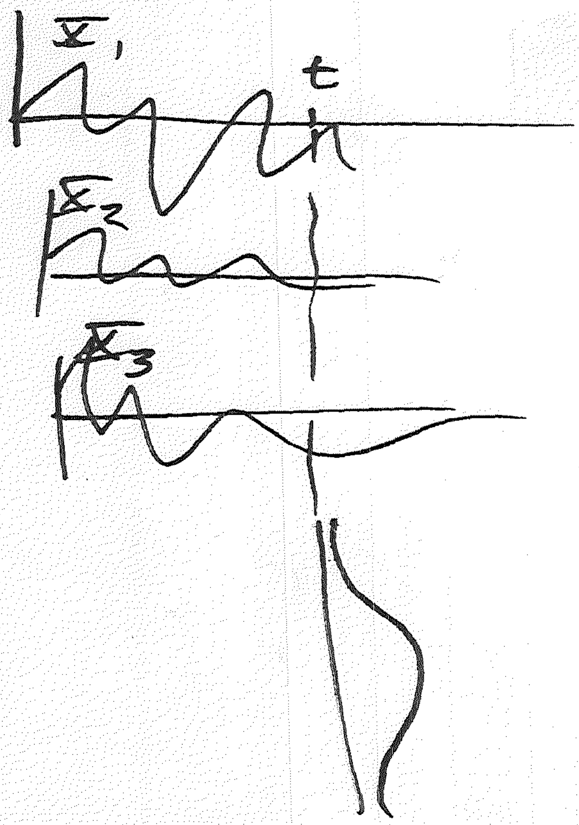
Set t.

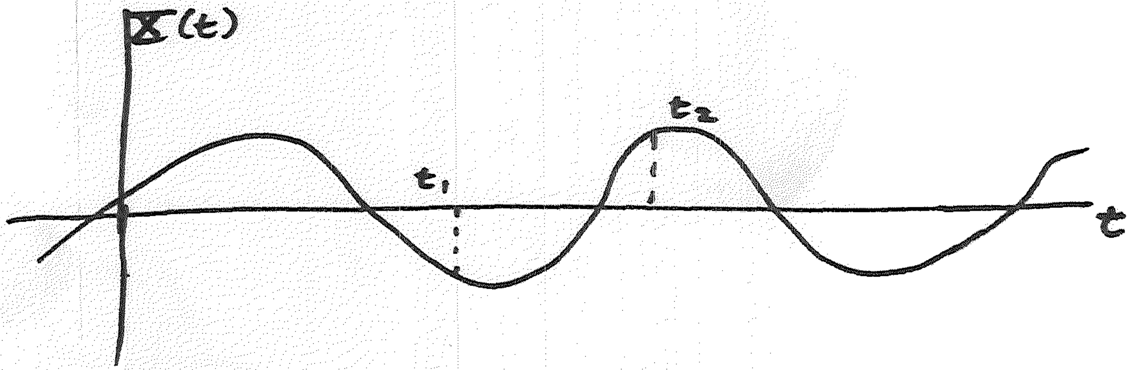
~~Define~~

Corresponding R.V. has pdf:

$$f(x;t) = \frac{\partial F(x,t)}{\partial x}$$

Ensemble (rel freq)





$X(t_1)$ is related (maybe) to $X(t_2)$.

Define:

second order distribution → $F(x_1, x_2; t_1, t_2) = P_r[X(t_1) \le x_1, X(t_2) \le x_2]$

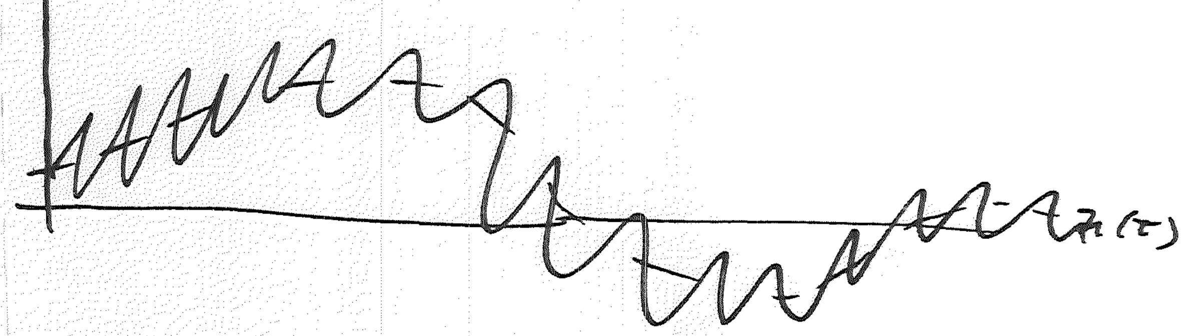
$$f(x_1, x_2; t_1\ t_2) = \frac{\delta^2}{\delta x_1 \delta x_2} F(x_1, x_2; t_1, t_2)$$

FIRST ORDER DISTR:

$$f_1(x_1, t_1) = \int_{x_2} f(x_1, x_2; t_1, t_2)\, dx_2$$

Moments:

mean $\eta(t) = E(x(t)) = \int_{-\infty}^{\infty} x\, f(x;t)\,dx$



Autocorrelation

$$R(t_1,t_2) = E[\mathbf{X}(t_1)\mathbf{X}(t_2)]$$

$$= \iint_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_1, t_2)\,dx_1 dx_2$$

AUTO COVARIANCE:

$$C(t_1,t_2) = E[(\mathbf{X}(t_1) - \eta(t_1))(\mathbf{X}(t_2) - \eta(t_2))]$$

$$= R(t_1,t_2) - \eta(t_1)\eta(t_2)$$

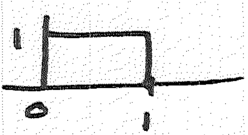
Note:

$$\sigma^2_{x(t)}(t) = C(t,t)$$

$$= R(t,t) - \eta^2(t)$$

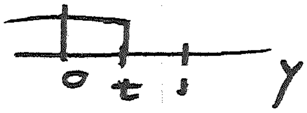
WRONG

Ex $X(t) = u(t - Y)$

$Y \sim$ 

$$\eta(t) = E(X(t)) = \int_0^1 u(t-y)\,dy$$

$$= \int_0^1 u(t-y)\,dy$$



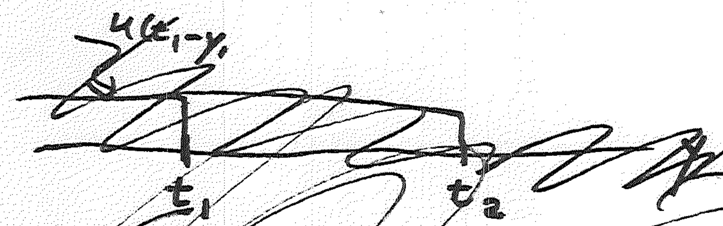
$$= \begin{cases} 0 & ; t < 0 \\ t & ; 0 \le t < 1 \\ 1 & ; t \ge 1 \end{cases}$$

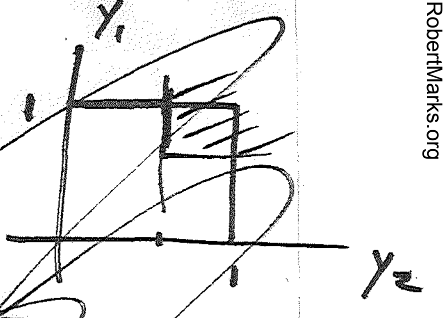
$$R(t_1, t_2) = E(X(t_1), X(t_2))$$

~~ASSUME Y_1 & Y_2 IND $\Rightarrow f_{Y_1 Y_2} = f_{Y_1} f_{Y_2}$~~

$$= \int_0^1 \int_0^1 u(t_1 - Y_1)\, u(t_2 - Y_2)\, dy_1\, dy_2$$

~~assume in~~



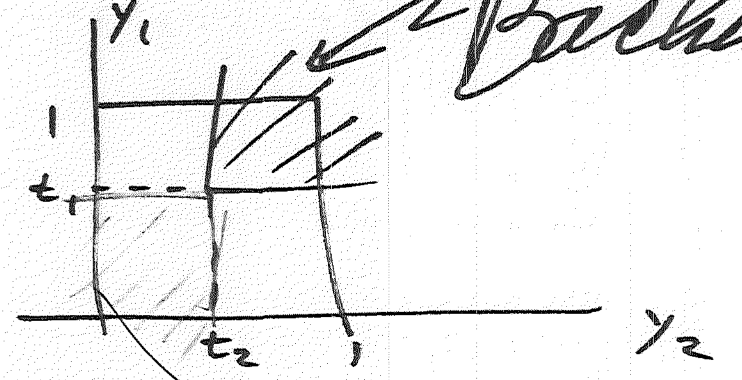


① $t_1 > 1,\ t_2 > 1 \Rightarrow R(t_1 t_2) = 0$

② $t_1 > 1,\ 0 < t_2 < 1 \Rightarrow R(t_1 t_2) = 1 - t_2$

③ $0 \le t_1 \le 1 \quad t > 1 \Rightarrow R(t_1, t_2) = 1 - t_1$

④ $0 \le t_1 \le 1;\ 0 \le t_2 \le 1 \Rightarrow R(t_1, t_2) = (1 - t_1)(1 - t_2)$



Backwards?

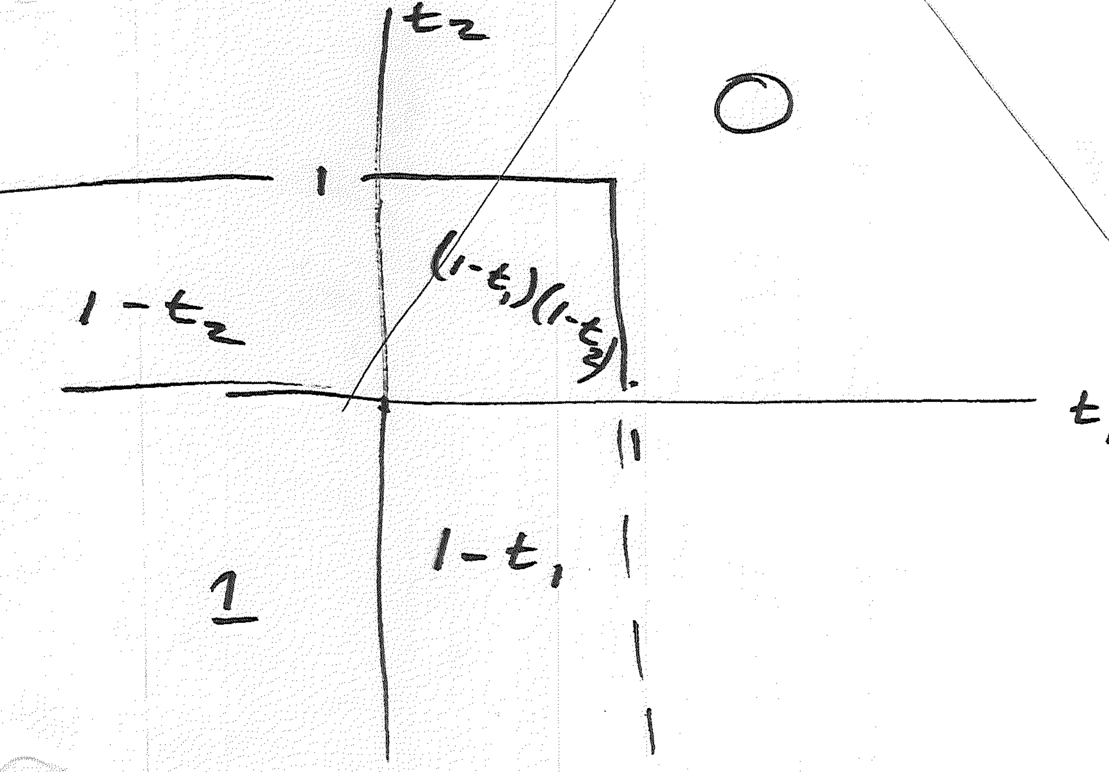
① $t_1 > 1$ or $t_2 > 1 \Rightarrow R(t_1, t_2) = 0$

② $0 \le t_1 < 1$, $t_2 < 0 \Rightarrow R(t_1\ t_2) = (1 - t_1)$

③ $t_1 < 0$; $0 < t_2 < 1 \Rightarrow R(t_1 t_2) = (1 - t_2)$

④ $0 < t_1 < 1$; $0 < t_2 < 1 \Rightarrow R(t_1 t_2) = (1-t_1)(1-t_2)$

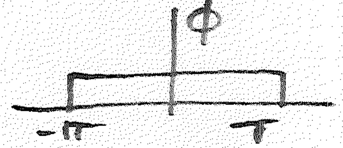
⑤ $t_1 < 0$, $t_2 < 0 \Rightarrow R(t_1 t_2) = 1$



EXAMPLE:

$$\mathbf{X}(t) = r \cos(\omega t + \Phi)$$

r & Φ IND r.v's



$$E[\mathbf{X}(t_1)\,\mathbf{X}(t_2)] = R(t_1, t_2)$$

$$= E\left[r^2 \cos(\omega t_1 + \phi) \cos(\omega t_2 + \phi)\right]$$

$$= \tfrac{1}{2} E[r^2]\, E\Big[\cos\omega(t_1 - t_2) - \cos(\omega t_1 + \omega t_2 - 2\phi)\Big]$$

But

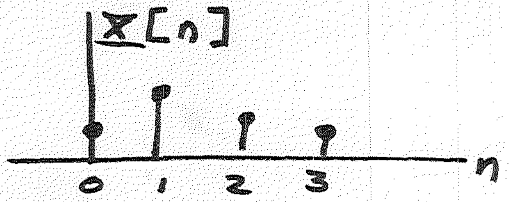
$$E[\cos(\omega t_1 + \omega t_2 - 2\phi)]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t_1 + \omega t_2 - 2\phi)\, d\phi = 0$$

Thus:

$$R(t, t_2) = \tfrac{1}{2} E[r^2] \cos\omega(t_1 - t_2)$$

Discrete Time Processes



$$\eta[n] = E[\mathbf{X}[n]]$$ mean

$$R[n_1, n_2] = E[\mathbf{X}[n_1]\,\mathbf{X}^*[n_2]]$$ autocorrelation

$$C[n_1, n_2] = R[n_1, n_2] - \eta[n_1]\,\eta^*[n_2]$$

autocovariance

White noise:

$$C[n_1, n_2] = q[n_1]\;\delta[n_1 - n_2]$$

$$q[n] = E[\mathbf{X}^2[n]]$$

$\mathbf{X}[n]$ is wss if

$$E[\mathbf{X}[n+m]\,\mathbf{X}^*[n]] = R[m]$$

Sampling

$$\mathbf{X}[n] = \mathbf{X}(nT)$$

$$\eta[n] = \eta(nT)$$

$$R[n_1, n_2] = R(n_1 T, n_2 T)$$

If stationary:

$$R[m] = R(mT)$$

Ex $x(t) = p + qt$

$p \sim f_p$ IND $q \sim f_q$

$\eta(t) = E[x(t)] = E[p + qt]$

$= E(p) + E(q)t$

$R(t_1, t_2) = E[(p + qt_1)(p + qt_2)]$

$= E(p^2) + E(pq)(t_1 + t_2) + E(q^2)t_1 t_2$

$C(t_1, t_2) = \sigma_p^2 + t_1 t_2 \sigma_q^2$

SPECIAL PROCESSES:

POISSON

PLACE n POINTS ON INTERVAL $(0, T)$

LET $t_2 - t_1 = t_a$

WHAT IS $Pr[k \in t_a]$?

$$= \binom{n}{k} p^k q^{n-k} \quad p = \frac{t_a}{T}$$

if $n >> 1$, $p = \frac{t_a}{T} << 1$

$$Pr[k \in t_a] \simeq e^{-n\frac{t_a}{T}} \frac{(n t_a/T)^k}{k!}$$

$$= e^{-\lambda t_a} \frac{(\lambda t_a)^k}{k!} \quad ; t_a = t_2 - t_1$$

~~IF $n \to \infty$, $T \to \infty$ $\frac{n}{T} \to \lambda$~~

$n \to \infty$, $T \to \infty$ $\frac{n}{T} \to \lambda$, BECOMES EXACT

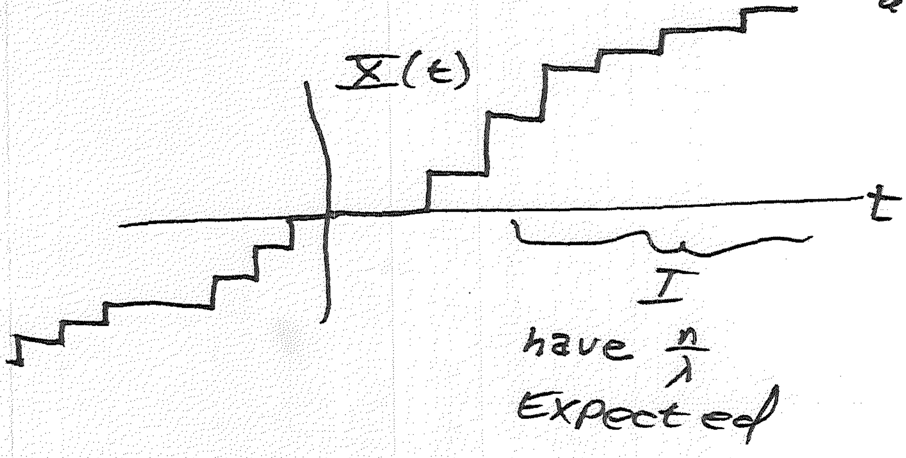
Define PRV PROCESS as follows:

Let $X(0) = 0$

$X(t)$ = # POINTS ON INTERVAL $(0, t)$

EQUIVALENTLY:

$X(t_a) - X(t_b)$ = # POINTS twixt t_a ; t_b



First order

STATISTICS:

$t_a > t_b$

$$P_r\left[X(t_a) - X(t_b) = k\right] = e^{-\lambda(t_a-t_b)} \frac{\left[\lambda(t_a-t_b)\right]^k}{k!}$$

Recall: $f_X(x) = e^{-a} \sum_{n=0}^{\infty} \frac{a^k}{k!} \delta(x-k) \quad ; a = \lambda(t_b - t$

$$E(x) = a$$

~~Var(x) =~~

~~E var x = a²~~

$$\Rightarrow E(x^2) = a^2 - a$$

$$\therefore E\left[X(t_a) - X(t_b)\right] = \lambda(t_a - t_b) \leftarrow \text{ELABORATE}$$

$$E\left[\left(X(t_a) - X(t_b)\right)^2\right] = \lambda^2(t_a-t_b)^2 + \lambda(t_a - t_b)$$

~~R[X(t_a) - X(t_b)~~

Second Order

For Poisson Process:

$$R(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

Proof: For $t_1 < t_2$

$\underline{X}(t_1)$ is ind. of $\underline{X}(t_2) - \underline{X}(t_1)$ if $t_1 < t_2$

(just like weiner)

$$E\left[\underline{X}(t_1)\left[\underline{X}(t_2) - \underline{X}(t_1)\right]\right]$$
$$= E\left[\underline{X}(t_1)\right] E\left[\underline{X}(t_2) - \underline{X}(t_1)\right]$$
$$= \lambda t_1 \cdot \lambda (t_2 - t_1)$$

Since:

$$\underline{X}(t_1)\underline{X}(t_2) = \underline{X}(t_1)\left[\underline{X}(t_1) + \underline{X}(t_2) - \underline{X}(t_1)\right]$$

We have

$$R(t_1, t_2) = E\left[\underline{X}(t_1)\underline{X}(t_2)\right]$$
$$= \overline{\underline{X}^2(t_1)} + \lambda^2 t_1 (t_2 - t_1)$$

$$\overline{\underline{X}^2(t_1)} = \text{second moment of Poisson RV}$$
$$= \lambda t_1 + \lambda^2 t_1^2$$

$$\Rightarrow R(t_1, t_2) = \lambda t_1 + \lambda^2 t_1^2 + \lambda^2 t_1 (t_2 - t_1) \;;\; t_1 < t_2$$
$$= \lambda t_1 + \lambda^2 t_1 t_2$$

for $t_1 > t_2$, switch. Q.E.D.

STATIONARY IN WIDE SENSE
(WEAKLY STATIONARY)

IF $E[X(t)] = \eta$

$E[X(t+\tau)X(\tau)] = R(\tau)$

STATIONARY PROCESSES

STRICT SENSE STATIONARITY

$X(t)$ IS STRICTLY STATIONARY IF IT HAS THE SAME STATISTICS AS $X(t+\epsilon)$

$X(t)$ & $Y(t)$ ~~HAVE~~ ARE JOINTLY STATIONARY IF SAME STATISTICS AS $X(t+\epsilon), Y(t+\epsilon)$

$X(t)$ & $Y(t)$ MIGHT BE INDIVIDUALLY, BUT NOT JOINTLY STATIONARY

FOR SR. STATIONARY R.V.

$$E[X(t)] = E[X(t+\epsilon)] \; \forall \epsilon$$

$$\therefore \eta(t) = \eta = \text{CONSTANT}$$

SINCE

$$f(x_1, x_2; t_1, t_2) = f(x_1, x_2; t_1+\epsilon_1, t_2+\epsilon)$$

$R(t_1,t_2) = E[\mathbf{X}(t_1)\, \mathbf{X}^*(t_2)]$ ~~$= E[\mathbf{X}(t_1)\mathbf{X}(t_1$~~

$$= R(t_1 - t_2) = R(\tau)$$

$$R(\tau) = E[\mathbf{X}(t+\tau)\, \mathbf{X}^*(t)] = R(-\tau), \mathbf{X} \text{ real}$$

CROSS CORRELATION:

$$R_{\mathbf{X}\mathbf{Y}}(\tau) = E[\mathbf{X}(t+\tau)\, \mathbf{Y}^*(t)]$$

~~Ex~~ $\mathbf{X}(t)$ is wss

$$S = \int_{-T}^{T} \mathbf{X}(t)\,dt$$

$$E[S] = 2T\eta \quad ; \eta = E[\mathbf{X}]$$

~~$\sigma_S^2 = E[(S-\eta)^2]$~~

E

$$E(s^2) = E\int_{-T}^{T}\int_{-T}^{T} \mathbf{X}(t_1)\,\mathbf{X}(t_2)\,dt_1\,dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} E[\mathbf{X}(t_1)\mathbf{X}(t_2)]\,dt_1\,dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} R(t_1,t_2)\,dt_1\,dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} R(t_2-t_1)\,dt_1\,dt_2$$

$$\tau = t_2 - t_1 \Rightarrow t_1 = t_2 - \tau$$

$$E[s^2] = \int_{-T}^{T}\int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t_2-\tau}{2T}\right) R(\tau)\,d\tau\,dt_2$$

$$= \int_{\tau=-\infty}^{\infty} R(\tau) \int_{\cancel{-T}\,-\infty}^{\cancel{T}\,\infty} \mathrm{rect}\left(\frac{t_2}{2T}\right)\mathrm{rect}\left(\frac{t_2-\tau}{2T}\right) dt_2\,d\tau$$

$$= \int_{\tau=-2T}^{2T} R(\tau)\,[2T - |\tau|]\,d\tau$$

Similarly

$$\sigma_S^2 = \int_{-2T}^{2T} (2T - |\tau|)\,C(\tau)\,d\tau$$

<u>Ex</u>

$a_i \sim i = 1, 2, \ldots, N$; a_i uncorrelated, $E(a_i) = 0$, $E[|a_i|^2] = c$

(~~Fourier~~ Trig polynomial / ~~Fourier~~ series special case)

$$\cancel{\text{Ex}} \quad X(t) = \sum_{n=1}^{N} a_i e^{j\omega_i t}$$

$$E[X(t)] = \cancel{\sum_{n=1}} \; 0$$

$$\cancel{E[X^2(t)] = E \sum_{n=1}^{N} \sum_{m=1}^{N} \sigma_i^2}$$

$$R(\tau) = E[X(t+\tau) X^*(t)]$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{N} \sigma_n^2 \cancel{\delta_n} e^{j\omega_n (t+\tau)} e^{-j\cancel{\omega}_n t}$$

$$= \sum_{n=1}^{N} \cancel{\sum_{m=1}^{N}} \sigma_n^2 e^{j\omega_n \tau} \Leftarrow \text{WIDE SENSE STATIONARY}$$

<u>~~Ex~~ Ex</u> a_i & b_i uncorrelated, zero mean

$$E[a_i^2] = E[b_i^2] = \sigma_i^2$$

$$X(t) = \sum_{i=1}^{N} (a_i \cos \omega_i t + b_i \sin \omega_i t)$$

Trig ~~polynomial~~ ~~Fourier~~ ~~series~~ special case

Let $c_i = a_i - jb_i$

$$Y(t) = \sum_{i=1}^{N} c_i e^{j\omega_i t}$$

$$X(t) = \mathrm{Re}\, Y(t)$$

Follows from prev. ex.:

$$R(\tau) = \mathrm{Re} \sum_{n=1}^{N} \sigma_n^2 e^{j\omega_n \tau}$$

$$= \sum_{n=1}^{N} \sigma_n^2 \cos \omega_n \tau$$

CONSIDER ST. R. PROCESS

$\eta_x = E[\underline{X}(t)]$ ← MEAN

$\underset{XX}{R}(\tau) = E[\underline{X}(t+\tau)\,\underline{X}^*(t)] = R_x^*(-\tau)$ ← AUTOCOR

$R_{XY}(\tau) = E[\underline{X}(t+\tau)\;\underline{Y}^*(\cancel{t})]$ ⇐ CROSS CORRELATION

~~NOTE~~

$\cancel{R_{XY}(\tau) = \iint_x x(t+\tau)\,y^*(t)\,f_{\underline{X}\underline{Y}}(x,y;t+\tau,\tau)\,dx\,dy}$

$\cancel{= \int_x\int_y x(t)\;y^*(t+\tau)}$

NOTE:

$$R_{XY}^*(\tau) = E[\underline{X}^*(t+\tau)\;\underline{Y}(t)]$$

$$= E[\underline{X}^*(t)\,\underline{Y}(t-\tau)] = R_{YX}(-\tau)$$

Ex

$z(t) = \underline{X}(t) + \underline{Y}(t)$

$$R_{zz}(\tau) = E[z(t+\tau)\;z^*(t)]$$

$$= E[(x(t+\tau) + y(t+\tau))(x^*(t) + y^*(t))]$$

$$= R_x(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$

Ex IF $X(t)$ & $Y(t)$ ARE IND

∋ $w = x(t)\,y(t)$

then

$$R_w(\tau) = R_x(\tau)\,R_y(\tau)$$

Thm : ~~If $E(x)=0$~~,

$$R(0) = E\{|x(t)|^2\} \geq 0$$

If not ~~$E[(x-\eta)] = E[x^2] - \eta^2 = R(0) - \eta^2 = \max$~~

IF $X(t)$ is Real!

$$E[\,|X(t+\tau) \pm X(t)|^2\} = E\left[\left(X(t+\tau) \pm X(t)\right)\left(X^*(t+\tau) \pm X^*(t)\right)\right]$$

$$= R(0) + R(0) \pm \left(R(\tau) + R^*(\tau)\right)$$

$$= 2R(0) \pm 2\,\mathrm{Re}\,R(\tau) \geq 0$$

$$(-) \Rightarrow R(0) \geq \cancel{R(\tau)}\ \mathrm{Re}\,R(\tau)$$

$$(+)\ R(0) \geq -\mathrm{Re}\,R(\tau)$$

or

$$|\mathrm{Re}\,R(\tau)| \leq R(0)$$

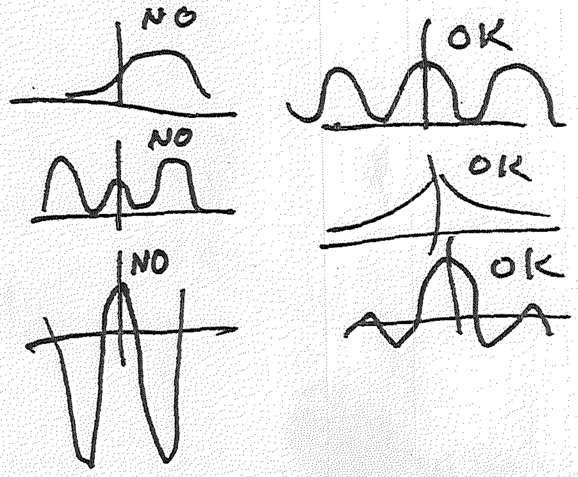
For real process

$$= 2\left[R(0) \pm R(\tau)\right] \geq 0$$

$$R(0) \geq R(\tau)$$

$$R(0) \leq -R(\tau)$$

$$\Rightarrow |R(\tau)| \leq R(0) \leftarrow \text{MAX}$$



Fourier Integrals of Stochastic Processes

$$\underset{\nwarrow}{X(\omega)} = \int_{-\infty}^{\infty} \underset{\nearrow}{x(t)}\, e^{-j\omega t}\, dt$$

Stochastic Processes

Inversion:
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Properties:

1. First order Statistics:
$$E[X(\omega)] = \int_{-\infty}^{\infty} \underbrace{E[x(t)]}_{\eta_x(t)}\, e^{-j\omega t}\, dt$$

~~$E\,X(\omega) = \mathcal{F}$~~

note: For stationary process, $E[X(\omega)] = \infty$ if $\eta \neq 0$

2. Second order statistics:

$$E[X(u) X^*(v)] = \Gamma(u, -v)$$

where
$$\Gamma(u,v) = \iint_{-\infty}^{\infty} R(t_1, t_2)\, e^{-j(ut_1 + vt_2)}\, dt_1 dt_2$$

Proof:
$$E[X(u) X^*(v)] = E \int_{t_1} x(t_1) e^{-jut_1} dt_1 \int_{t_2} x^*(t_2) e^{+jvt_2} dt_2$$
$$= \iint_{t_1 t_2} R(t_1, t_2)\, e^{-j(ut_1 + (-v)t_2)}\, dt_1 dt_2$$

3. If $x(t)$ is WSS with power spectrum $S(\omega)$, then $X(\omega)$ is white noise with average intensity $2\pi S(u)$:

$$E[X(u)X^*(v)] = 2\pi\, S(u)\, \delta(u-v)$$

Proof: If ~~Rt~~ WSS, $R(t_1,t_2) = R(t_1-t_2)$

$$\Gamma(u,v) = \iint_{-\infty}^{\infty} R(t_1-t_2)\, e^{-j(ut_1+vt_2)}\, dt_1\, dt_2$$

$$\tau = t_1 - t_2$$

$$= \int_{-\infty}^{\infty} e^{-j(u+v)t_2} \int R(\tau)\, e^{-ju\tau}\, d\tau\, dt_2$$

$$= S(u) \int_{-\infty}^{\infty} e^{-j(u+v)t_2}\, dt_2$$

$$= S(u)\; 2\pi\; \delta(v+u)$$

4. If $x(t)$ is white:

$$R(t_1,t_2) = q(t_1)\,\delta(t_2-t_1)$$

then $X(\omega)$ is WSS with autocorrelation:

$$E[X(u)X^*(v)] = Q(u-v)$$

where

$$Q(\omega) = \int_{-\infty}^{\infty} q(t)e^{-j\omega t}dt$$

Proof:

$$\Gamma(u,v) = \iint q(t_1)\,\delta(t_1-t_2)\,e^{-j(ut_1+vt_2)}dt_1dt_2$$

$$= \int q(t_1)e^{-j(u+v)t_2}dt_2$$

$$= Q(u+v)$$

Note:

$$E[|X(\omega)|^2] = Q(0) = \int_{-\infty}^{\infty} q(t)dt$$

(Look@ 10:34 for final)

POWER SPECTRUM

S(ω) = POWER SPECTRUM
(SPECTRAL DENSITY)

$$= \int_{-\infty}^{\infty} e^{-j\omega\tau} R(\tau)\, d\tau$$

SINCE $R(\tau) = R^*(-\tau)$ ← (HERMETIAN)

S(ω) is Real Function ≥ 0

INVERSION

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{j\omega\tau} d\omega$$

NOTE:

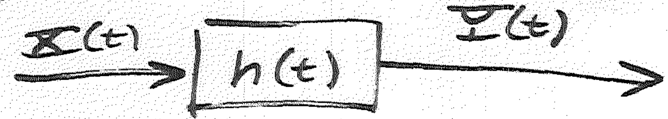
$$R(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = E\{|\underline{x}(t)|^2\}$$

EQUIVALENTLY:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\cos\omega\tau\, d\tau \quad \text{if rea}$$

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\cos\omega\tau\, d\omega$$

LINEAR SYSTEMS:
(Review)



$$Y(t) = X(t) * h(t)$$

$$= \int_{-\infty}^{\infty} X(\tau) h(t-\tau)$$

ASSUME X IS STATIONARY:

$$\eta_Y = E[Y(t)] = \int_{-\infty}^{\infty} E[X(t-\tau)]\, h(\tau) d\tau$$

$$= \eta_x \int_{-\infty}^{\infty} h(\tau) d\tau$$

$$= \eta_x H(0)$$

AUTOCORRELATION:

$$R_Y(\tau) = E[Y^*(t)\, Y(t+\tau)]$$

$$= \cancel{E\left[\int_{-\infty}^{\infty} X(\alpha) h(t-\alpha) d\alpha \int_{-\infty}^{\infty} \overset{X(t+}{X(\beta)} h(t+\tau-\beta) d\beta\right.}$$

$$\cancel{\int_{-\infty}^{\infty}}$$

$$= \cancel{\iint_{-\infty}^{\infty} X(\alpha) X(\beta)} = E[$$

CONSIDER:

$$y(t)\, x^*(t-\tau) = \int X(t-\alpha) X^*(t-\tau)\, h(\alpha) d\alpha$$

$$R_{yx}(\tau)\, E[\quad] = \int R_x(\tau-\alpha)\, h(\alpha) d\alpha$$

$$R_{yx}(\tau) = R_x(\tau) * h(\tau)$$

$$y(t+\tau)\,y^*(t) = \int_{-\infty}^{\infty} y(t+\tau)\,x^*(t-\alpha)\,h^*(\alpha)\,d\alpha$$

$$R_y(\tau) = \int_{-\infty}^{\infty} R_{yx}(\tau+\alpha)\,h^*(\alpha)\,d\alpha$$

$$= R_{yx}(\tau) * h^*(-\tau)$$

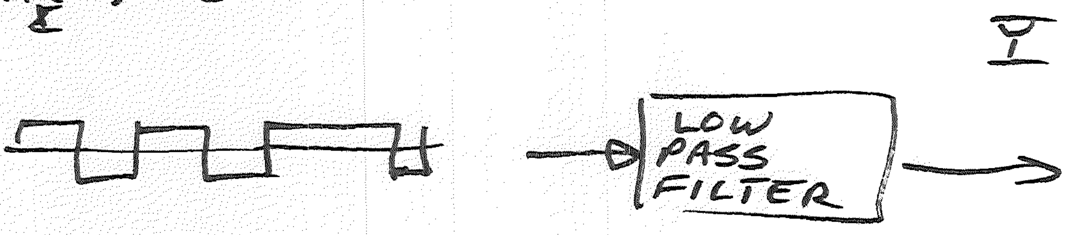
$$= R_x(\tau) * \left(\underbrace{h(\tau) * h^*(-\tau)}_{h \star h}\right)$$

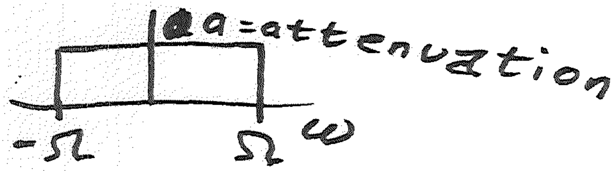
$$S_y(\omega) = S_x(\omega)\,\left|H(j\omega)\right|^2$$

Ex:

Random Telegraph Signal
(Randomized Origin)

$R_{\underline{X}}(\tau) = e^{-2\lambda|\tau|}$





$$S_{\underline{Y}}(\omega) = a^2 \frac{4\lambda}{4\lambda^2+\omega^2}\, \text{rect}\left(\frac{\omega}{2\Omega}\right)$$

$$R_{\underline{Y}}(\tau) = \frac{4a^2\lambda}{2\pi}\int_{-\Omega}^{\Omega}\frac{e^{j\omega\tau}\,d\omega}{4\lambda^2+\omega^2}$$

$$\text{var}\,\underline{Y} = R_{\underline{Y}}(0)$$

$$= \frac{2a^2\lambda}{\pi}\int_{-\Omega}^{\Omega}\frac{d\omega}{4\lambda^2+\omega^2}$$

$$= \frac{\cancel{4}a^2\lambda}{\cancel{4}\lambda^2\pi}\int_0^{\Omega}\frac{d\omega}{1+\left(\frac{\omega}{2\lambda}\right)^2}$$

$$\hat{\omega} = \frac{\omega}{2\lambda}$$

$$= \frac{a^2\cancel{\lambda}}{\cancel{\lambda}\pi}\int_0^{\frac{\Omega}{2\lambda}}\frac{d\hat{\omega}\, 2\cancel{\lambda}}{1+\hat{\omega}^2}$$

$$= \frac{2a^2}{\pi}\,\text{atan}\frac{\Omega}{2\lambda}$$

$$\xrightarrow[\substack{\Omega\to\infty \\ a\to 1}]{} \frac{2}{\pi}\cdot\frac{\pi}{2} = \underline{1}$$

Differentiation of $X(t)$

Fourier: $\mathcal{F}\frac{d}{dt}x(t) = (j\omega)F(\omega)$

From relation

$$S_{X'}(\omega) = |j\omega|^2 S_X(\omega)$$
$$= \omega^2 S_X(\omega)$$
$$= (-1)(j\omega)^2 S_X(\omega)$$
$$\Rightarrow R_{X'}(\tau) = -\left(\frac{d}{d\tau}\right)^2 R_X(\tau) \longrightarrow$$

Mean

$$\eta_{X'} = \frac{d}{dx}\eta_X$$

We assume ~~but~~ R_X can be twice differentiated.

O.W., process deriv. does ~~is~~ not exist

(Ex: Telegraph signal)

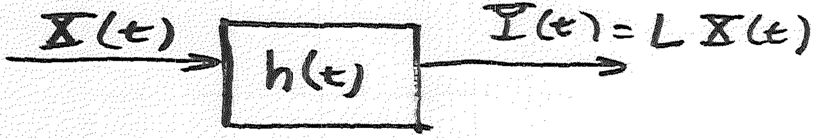
Generalizing

$$R_{X^{(n)}}(\tau) = (-1)^n \left(\frac{d}{d\tau}\right)^{2n} R_X(\tau)$$

if. $\left(\frac{d}{d\tau}\right)^{2n}$ exists.

$$\to \eta_{X^{(n)}} = \left(\frac{d}{dt}\right)^n \eta(t)$$

Linear Time Invariant System Response to non-stationary inputs



First order statistics:

Fundemental Theorem: $E\,L\,\mathbf{X} = L\,E\,\mathbf{X}$

~~Thus:~~ Proof:

$$\eta_y(t) = E[\mathbf{Y}(t)] = E\int_{-\infty}^{\infty} \mathbf{X}(t-\alpha)\,h(\alpha)\,d\alpha$$
$$= \int_{-\infty}^{\infty} \eta_x(t-\alpha)\,h(\alpha)\,d\alpha$$
$$= \eta_x(t) * h(t)$$

Second order statistics

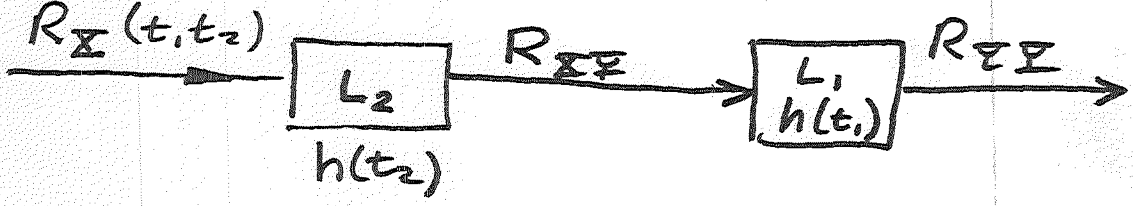
Theorem: $R_{\mathbf{XY}}(t_1,t_2) = L_2\, R_{\mathbf{XX}}(t_1,t_2)$

L_2 OPERATES ON t_2

&

$R_{\mathbf{YY}}(t_1,t_2) = L_1\, R_{\mathbf{XY}}(t_1,t_2)$

~~ie~~ flow:



ie

$$R_{\mathbf{XY}} = R_{\mathbf{X}} \underset{t_2}{*} \cancel{R_{\mathbf{X}}}\, h(t_2)$$
$$R_{\mathbf{YY}} = R_{\mathbf{XY}} \underset{t_1}{*} h(t_1)$$

Proof:

$$y(t_2) = L_{t_2}\, x(t_2)$$

$$x(t_1)\, y(t_2) = x(t_1)\, L_{t_2}\, x(t_2)$$

$$= L_{t_2}\, x(t_2)\, x(t_1)$$

E both sides:

$$R_{\mathbf{X}\mathbf{Y}}(t_1, t_2) = L_{t_2}\, R_{\mathbf{X}}(t_1, t_2)$$

End of Part 1

$$y(t_1) = L_1\, x(t_1)$$

$$y(t_2)\, y(t_1) = L_1\, x(t_1)\, y(t_2)$$

$$R_{\mathbf{Y}}(t_1, t_2) = L_1 R_{\mathbf{X}\mathbf{Y}}(t_1, t_2)$$

$R_{\mathbf{Y}}$ in terms of $R_{\mathbf{X}}$

$$R_{\mathbf{X}\mathbf{Y}} = R_{\mathbf{X}} \underset{t_2}{*} h$$

$$R_{\mathbf{Y}} = R_{\mathbf{X}\mathbf{Y}} \underset{t_1}{*} h$$

$$= R_{\mathbf{X}} \underset{t_1}{*} h \underset{t_2}{*} h$$

$$= R_{\mathbf{X}} \underset{2\text{-}D}{\underline{**}}\ h(t_1)\, h(t_2)$$

$$= \iint_{-\infty}^{\infty} R_{\mathbf{X}}(t_1 - \alpha, t_2 - \beta)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta$$

Similarly:

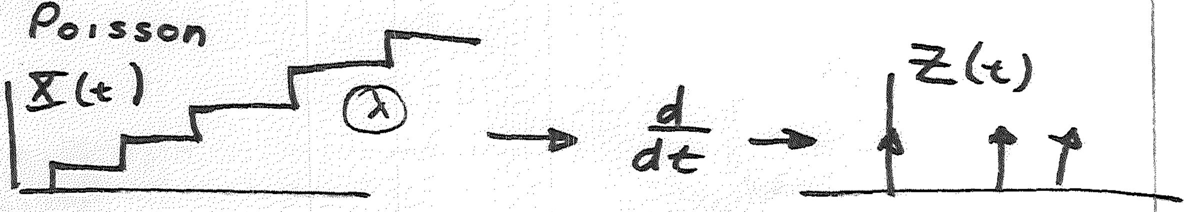
$$C_{\mathbf{X}\mathbf{Y}}(t_1, t_2) = C_{\mathbf{X}}(t_1, t_2) \underset{t_2}{*} h(t_2)$$

$$C_{\mathbf{Y}}(t_1, t_2) = C_{\overline{XY}}(t_1, t_2) \underset{t_1}{*} h(t_1)$$

Reduce to Station Case for $R(\tau)$

(~~Needs~~ more)

Ex



$$Z(t) = \sum_i \delta(t - t_i)$$

$$\eta_Z = \lambda \quad , \quad R_Z(\tau) = \lambda^2 + \lambda\,\delta(\tau)$$

Proof

Recall $\quad \eta_X(t) = \lambda t$

$$\Rightarrow \eta_Z = \frac{d}{dt}\lambda t = \lambda$$

Recall:

$$R_{XX}^{(t_1 t_2)} = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

$$R_{XZ}(t_1 t_2) = \frac{\delta R_{XX}(t_1 t_2)}{\delta t_2}$$

$$= \lambda^2 t_1 + \lambda\, U(t_1 - t_2)$$

$$R_{ZZ}(t_1 t_2) = \frac{\delta R_{XZ}(t_1 t_2)}{\delta t_1}$$

$$= \lambda^2 + \lambda\,\delta(t_1 - t_2)$$

Ex: Non-stationary white noise

~~$x(t) =$~~

$$R_{\mathbf{X}}(t_1, t_2) = q(t_1)\,\delta(t_1 - t_2)$$

(for stat. white, $q(t_1) = \text{const}$).

$$R_{\mathbf{XY}} = R_{\mathbf{X}} \underset{t_2}{*} h(t_2)$$

$$= q(t_1)\;\delta(t_2 - t_1) * h(t_2)$$

$$= q(t_1)\, h(t_2 - t_1)$$

$$R_{\mathbf{Y}} = R_{\mathbf{XY}} \underset{t_1}{*} h(t_1)$$

$$= \int_{-\infty}^{\infty} h(t_1 - \alpha)\, q(\alpha)\, h(t_2 - \alpha)\, d\alpha$$

Stochastic Diff. eq.
const coeff:

$$\sum_{i=0}^{n} a_n \underline{Y}^{(n)}(t) = \underline{X}(t)$$

If $\underline{X}$ is stationary, $\underline{Y}$ need not be

Given: $\underline{Y}^{(i)}(0) = 0 \quad ; i = 0 \ldots n-1$

Expectation:

$$\sum_{i=0}^{n} a_n \eta_n^{(i)}(t) = \eta_X(t)$$

$$\eta^{(i)}(0) = 0 \quad \left(\text{since } \underline{Y}^{(i)}(0) = 0\right)$$

Autocorrelation:

$$\sum_{i=0}^{n} a_n \underline{Y}^{(n)}(t_2) = \underline{X}(t_2)$$

$$\sum_{i=0}^{n} a_n \underline{X}(t_1) \underline{Y}^{(n)}(t_2) = \underline{X}(t_1)\underline{X}(t_2)$$

But: $$E[\underline{X}(t_1)\underline{Y}^{(i)}(t_2)] = \frac{d^i R_{XY}(t_1, t_2)}{dt_2^i}$$

Proof:

$$E\left[\underline{X}(t_1)\left(\frac{d}{dt_2}\right)^i \underline{Y}(t_2)\right]$$

$$= \left(\frac{d}{dt_2}\right)^i E[\underline{X}_1)\underline{Y}(t_2)]$$

Thus:

$$\sum_{i=0}^{n-1} a_n \frac{\delta^i R_{XY}(t_1, t_2)}{\delta t_2^i} = R_X(t_1, t_2)$$

Since:

$$x(t_1)\underline{Y}^{(i)}(0) = 0$$

$$\frac{d^i R(t_1, t_2)}{dt_2^i} = 0 \quad ; i = 1 \ldots n$$

Similarly:

$$\sum_{i=1}^{n} a_i \frac{\delta^n R_{YY}(t_1,t_2)}{\delta t_1^n} = R_{XX}(t_1,t_2)$$

$$\frac{\delta^i R_{YY}(0,t_2)}{dt_1^i} = 0$$

Similar
(Proof)

Ex $\frac{dY}{dt} + \alpha Y(t) = X(t)$; $Y(0)=0$

X is Stat:

$E X = \lambda$; $R_{XX}(\tau) = \lambda^2 + \lambda\, \delta(\tau)$

X ~ sequence of poisson pulses

$\eta_y'(t) + \alpha \eta_y(t) = \lambda$; $\eta(0) = 0$

$\Rightarrow \eta_y(t) = \frac{\lambda}{\alpha}(1 - e^{-\alpha t})$

Cross:
Correlation

$$\frac{\delta R_{XY}(t_1,t_2)}{\delta t_2} + \alpha R_{XY}(t_1,t_2) = \lambda^2 + \lambda\delta(t_1 - t_2) \; ; \quad R_{XX}(t_1-t_2)$$

Solution:

$$R_{XY}(t_1,t_2) = \frac{\lambda^2}{\alpha}(1 - e^{-\alpha t_2}) + \lambda e^{-\alpha(t_2-t_1)} \mu(t_2 - t_1)$$

Autocorrelation: (for $t_1 < t_2$)

$$\frac{\delta R_{YY}(t_1,t_2)}{\delta t_1} + \alpha R_{YY}(t_1,t_2) = \frac{\lambda^2}{\alpha}(1 - e^{-\alpha t_2}) + \lambda e^{-\alpha(t_2 - t_1)}$$

$R_{YY}(0, t_2) = 0$

Treat t_2 as const. Gives, for $t_2 > t_1$:

$$R_{YY}(t_1,t_2) = \frac{\lambda^2}{\alpha^2}(1 - e^{-\alpha t_2})(1 - e^{-\alpha t_1}) + \frac{\lambda}{2\alpha} e^{-\alpha(t_2 - t_1)} (1 - e^{-2\alpha t_1})$$

Otherwise $(t_2 < t_1)$, $R_{YY}(t_1,t_2) = R_{YY}(t_2, t_1)$

Ergodicity:

Preliminaries

$$s = \frac{1}{2T}\int_{-T}^{T} \overset{\text{wss}}{W(t)}\,dt$$

$$E[s] = \eta = E[W(t)]$$

$$\cancel{\overline{s^2} = \frac{1}{2T}\int_{-2T}^{2T}(2T-|\tau|)}$$

$$\overline{s^2} = \frac{1}{(2T)^2}\int_{-2T}^{2T}\left(1 - \frac{|\tau|}{2T}\right)R(\tau)\,d\tau$$

$$\text{var } s = \frac{1}{2T}\int_{-2T}^{2T}(2T-|\tau|)R(\tau)\,d\tau - \eta^2$$

$$= \frac{1}{2T}\int_{-2T}^{2T}(2T-|\tau|)\,[\underbrace{R(\tau)-\eta^2}_{C(\tau)}]\,d\tau$$

$$= \frac{1}{T}\int_{0}^{2T}(2T-|\tau|)\,(R(\tau)-\eta^2)\,d\tau$$

Ex: For telegraph (randomized origin)

$$R(\tau) = e^{-2\lambda|\tau|} \quad ; \eta = 0$$

$$\text{var } s = \frac{1}{2\lambda T} - \frac{1-e^{-2\lambda T}}{8\lambda^2 T^2}$$

TIME AVERAGES

$$\underset{\underset{\text{STATISTIC}}{\uparrow}}{\langle \underline{X} \rangle} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \underline{X}(t)\,dt$$

$$\langle R(\tau)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \underline{X}(t+\tau)\,\underline{X}(t)\,dt$$

$$\langle \sigma_{\underline{X}}^2 \rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left[\underline{X}(t) - \langle \underline{X} \rangle\right]^2 dt$$

~~Note~~ Ergodicity in the mean. Requires two criteria

1. $E[\langle \underline{X} \rangle] = \eta$
2. $\text{var}\,\langle \underline{X} \rangle = 0$

Then, η can be ~~estimated~~ determined with ~~probabilit~~ one stochastic process.

Equivalently:

1. $\displaystyle \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \underline{X}(t)\,dt = E[\underline{X}] = \eta \leftarrow$ TRUE IF WSS

2. $\displaystyle \lim_{T\to\infty} \frac{1}{T}\int_{0}^{2T} \left(1 - \frac{\tau}{2T}\right)\left[R(\tau) - \eta^2\right] d\tau = 0$

For telegraph:

1. WSS
2. $\displaystyle \frac{1}{\alpha T} - \frac{1-e^{-2\lambda T}}{8\lambda^2 T^2} \xrightarrow[T\to\infty]{} 0$

$\therefore$ Ergodic in the mean

Sufficient Condition for Ergodicity in the mean:

1. if $X(t)$ is WSS and $\int_{-\infty}^{\infty}|C(\tau)|d\tau<\infty$,
   X is mean ergodic

   Proof:

$$\frac{1}{2T}\int_{-2T}^{2T} C(\tau)\left(1-\frac{|\tau|}{2T}\right)d\tau$$

$$< \frac{1}{2T}\underbrace{\int_{-2T}^{2T}|C(\tau)|d\tau}_{\text{if finite, then} \xrightarrow[T\to\infty]{} 0}$$

2. If $C(0)<\infty$ & $C(\tau)\xrightarrow[|\tau|\to\infty]{}\cancel{\infty}\,0$

   then $X(t)$ is mean-ergodic

   Note: $X(t+\tau)$ & $X(t)$ are uncorrelated for $\tau\to\infty$

   Proof in text.

<u>Ex</u> $X(t) = A$

~~$x(t)$~~

$$\frac{1}{2T}\int_{-T}^{T} X(t)dt = A \neq E[A]$$

for all T

<u>not</u> mean ergodic.

Ex Stationary White Noise

$$C(t_1, t_2) = q\,\delta(t_1 - t_2)$$

or

$$C(\tau) = q\,\delta(\tau)$$

If zero mean

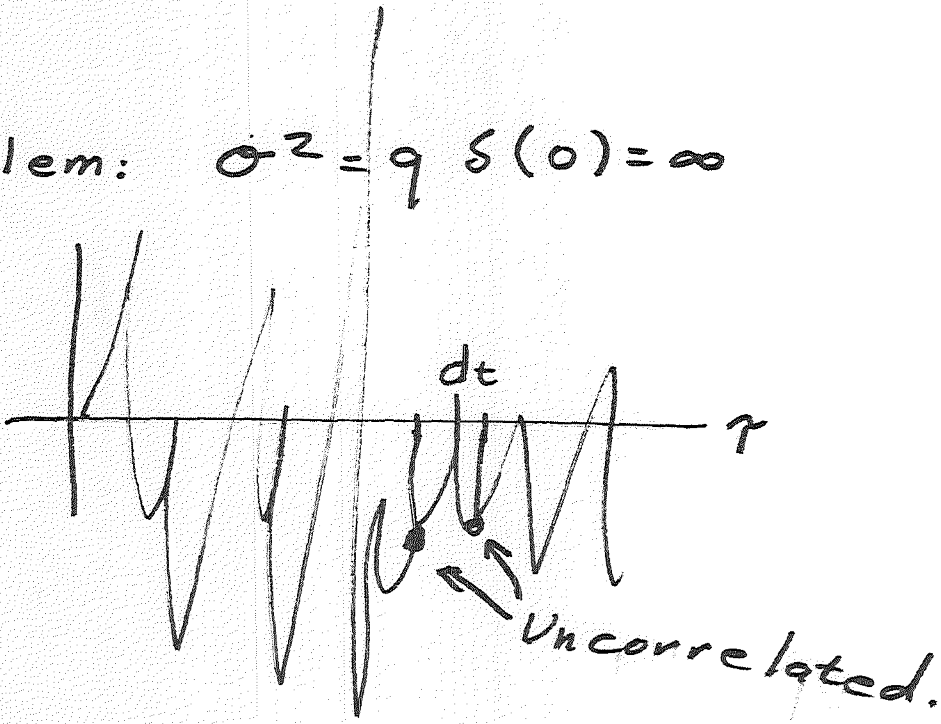
$$S(\omega) = q \Leftarrow \text{why called white}$$

From condition 1:

$$\int_{-\infty}^{\infty} |C(\tau)|\,d\tau = q < \infty$$

$\therefore$ ~~nois~~ process is mean ergodic

Problem: $\sigma^2 = q\,\delta(0) = \infty$



Correlation - Ergodic Process

$$\langle R(\tau)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t+\tau)\, X(t)\, dt$$

~~Define~~

Note

$$E\langle R(\tau)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E\, X(t+\tau)\, X(t)\, dt$$

$$= R(\tau)$$

Define

$$Z_\tau(t) = X(t+\tau)\, X(t)$$

If $Z_\tau(t)$ is mean-ergodic, then $X(t)$ is Correlation Ergodic.

(Requires fourth-order statistics).

Generating Random Numbers

1. Use Table
2. Pseudo-Random numbers

Congruence Method of generating ~~pse~~ pseudo-random numbers

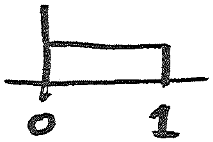
$$X_{n+1} = (a X_n + b)_{\text{MOD } T}$$

b & T should be relatively prime

Example: $a = \frac{3}{2}$, $b = \pi$, $T = 1$

(R) Seed $X_0 = 1$ ⟸ NOTE: CAN GET FROM RANDOM # TABLE

$$\left.\begin{aligned} X_1 &= 0.1415926 \\ X_1 &= 0.641592654 \\ X_2 &= 0.103981635 \\ X_3 &= 0.297565 \end{aligned}\right\} \text{Uniform}$$



Can Show:

$$\rho_s = E[X_n X_{n+s}]$$

$$= \frac{1 - 6 \frac{b_s}{T}\left(1 - \frac{b_s}{T}\right)}{a_s} + \epsilon$$

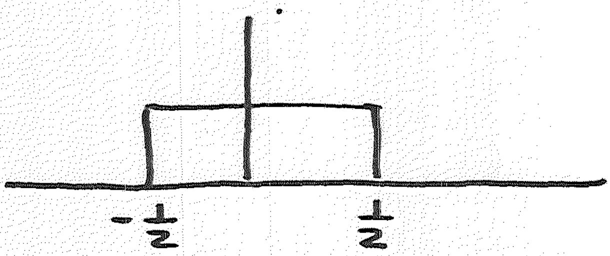
$$a_s = a^s \quad (\text{mod } T)$$

$$b_s = \left(\sum_{n=0}^{s-1} a^n\right) b \qquad (\text{mod } T)$$

$$|e| < a_s / T$$

HP $\quad Y_{n+1} = \text{Frc}(X_n + \pi)^5$

1. How can we generate



A: $Y = X - \frac{1}{2}$

2. What about a dice roll?

$$D_n = \text{Int}\left[6X_n + 1\right]$$

~~3. How about the sum of two dice?~~

~~$D_n + D_{n+1}$ = sum~~

3. Gaussian R.V.

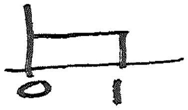
(a) can find the $g \ni Y_n = g(X_n)$ is gaussian (ugly!)

(b) Central limit theorem

$$Y = X_1 + X_2 + X_3 + \dots + X_N$$

$$\simeq \text{Mean} = \frac{N}{2}, \quad \text{var} = \frac{N}{12}$$

(c) Compute X_n & X_{n+1}



then

$$Y_n = (-2\ln X_1)^{\frac{1}{2}} \cos 2\pi X_1$$

$$Y_{n+1} = (-2\ln X_1)^{\frac{1}{2}} \sin 2\pi X_2$$

Y_n & Y_{n+1} are zero mean -unit variance normal r.v.'s.

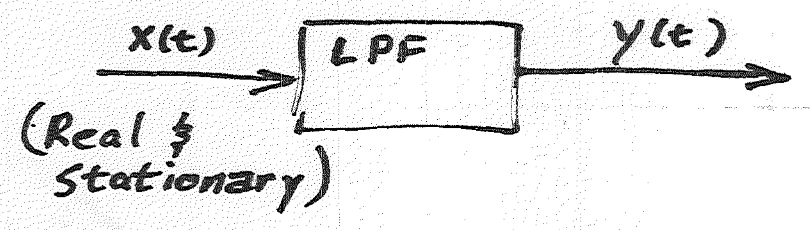
2500 FIVE DIGIT RANDOM NUMBERS Table 26.11

53479	81115	98036	12217	59526	40238	40577	39351	43211	69255
97344	70328	58116	91964	26240	44643	83287	97391	92823	77578
66023	38277	74523	71118	84892	13956	98899	92315	65783	59640
99776	75723	03172	43112	83086	81982	14538	26162	24899	20551
30176	48979	92153	38416	42436	26636	83903	44722	69210	69117
81874	83339	14988	99937	13213	30177	47967	93793	86693	98854
19839	90630	71863	95053	55532	60908	84108	55342	48479	63799
09337	33435	53869	52769	18801	25820	96198	66518	78314	97013
31151	58295	40823	41330	21093	93882	49192	44876	47185	81425
67619	52515	03037	81699	17106	64982	60834	85319	47814	08075
61946	48790	11602	83043	22257	11832	04344	95541	20366	55937
04811	64892	96346	79065	26999	43967	63485	93572	80753	96582
05763	39601	56140	25513	86151	78657	02184	29715	04334	15678
73260	56877	40794	13948	96289	90185	47111	66807	61849	44686
54909	09976	76580	02645	35795	44537	64428	35441	28318	99001
42583	36335	60068	04044	29678	16342	48592	25547	63177	75225
27266	27403	97520	23334	36453	33699	23672	45884	41515	04756
49843	11442	66682	36055	32002	78600	36924	59962	68191	62580
29316	40460	27076	69232	51423	58515	49920	03901	26597	33068
30463	27856	67798	16837	74273	05793	02900	63498	00782	35097
28708	84088	65535	44258	33869	82530	98399	26387	02836	36838
13183	50652	94872	28257	78547	55286	33591	61965	51723	14211
60796	76639	30157	40295	99476	28334	15368	42481	60312	42770
13486	46918	64683	07411	77842	01908	47796	65796	44230	77230
34914	94502	39374	34185	57500	22514	04060	94511	44612	10485
28105	04814	85170	86490	35695	03483	57315	63174	71902	71182
59231	45028	01173	08848	81925	71494	95401	34049	04851	65914
87437	82758	71093	36833	53582	25986	46005	42840	81683	21459
29046	01301	55343	65732	78714	43644	46248	53205	94868	48711
62035	71886	94506	15263	61435	10369	42054	68257	14385	79436
38856	80048	59973	73368	52876	47673	41020	82295	26430	87377
40666	43328	87379	86418	95841	25590	54137	94182	42308	07361
40588	90087	37729	08667	37256	20317	53316	50982	32900	32097
78237	86556	50276	20431	00243	02303	71029	49932	23245	00862
98247	67474	71455	69540	01169	03320	67017	92543	97977	52728
69977	78558	65430	32627	28312	61815	14598	79728	55699	91348
39843	23074	40814	03713	21891	96353	96806	24595	26203	26009
62880	87277	99895	99965	34374	42556	11679	99605	98011	48867
56138	64927	29454	52967	86624	62422	30163	76181	95317	39264
90804	56026	48994	64569	67465	60180	12972	03848	62582	93855
09665	44672	74762	33357	67301	80546	97659	11348	78771	45011
34756	50403	76634	12767	32220	34545	18100	53513	14521	72120
12157	73327	74196	26668	78087	53636	52304	00007	05708	63538
69384	07734	94451	76428	16121	09300	67417	68587	87932	38840
93358	64565	43766	45041	44930	69970	16964	08277	67752	60292
38879	35544	99563	85404	04913	62547	78406	01017	86187	22072
58314	60298	72394	69668	12474	93059	02053	29807	63645	12792
83568	10227	99471	74729	22075	10233	21575	20325	21317	57124
28067	91152	40568	33705	64510	07067	64374	26336	79652	31140
05730	75557	93161	80921	55873	54103	34801	83157	04534	81368

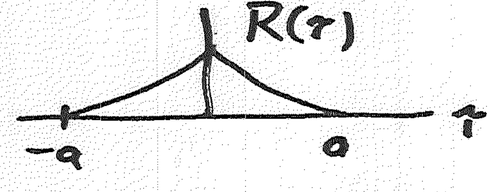
Compiled from Rand Corporation, A million random digits with 100,000 normal deviates. The Free Press, Glencoe, Ill., 1955 (with permission).

ANALOG TECHNIQUES

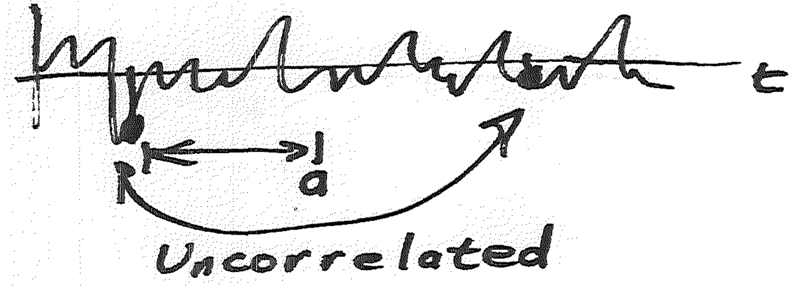
Low Pass Response



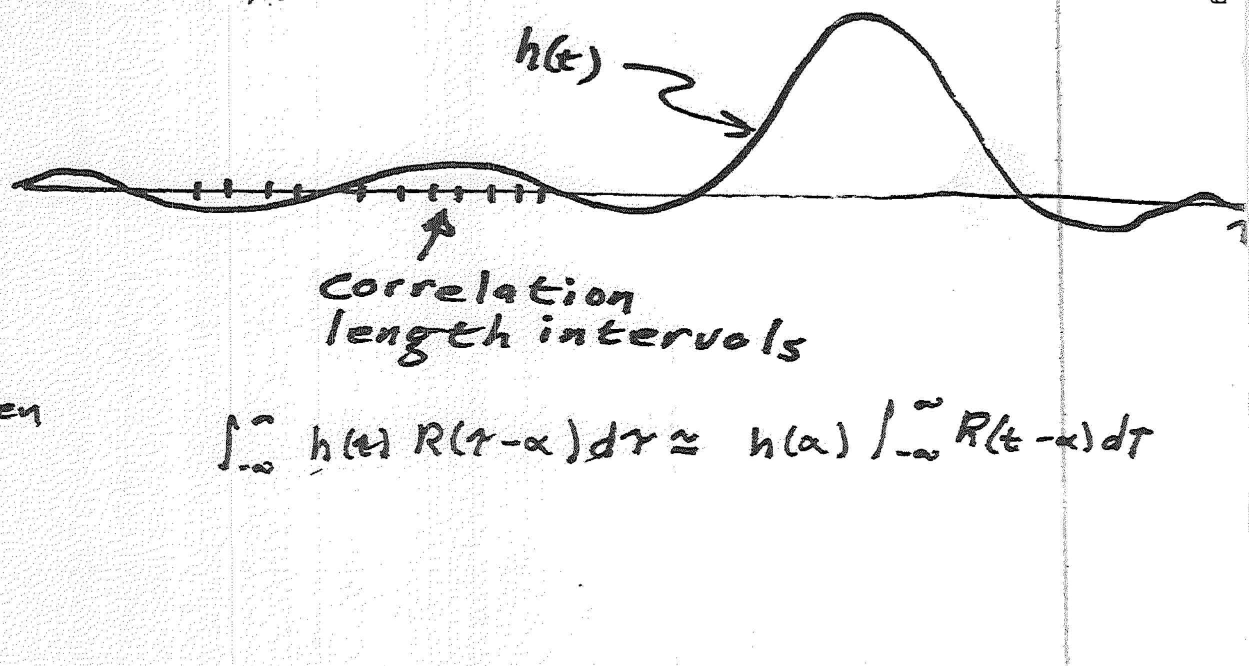
Assume $R(\tau) \simeq 0$ for $|\tau| > a$



a = correlation length



Also, assume $h(t) \simeq$ constant in any correlation length interval:



Then

$$\int_{-\infty}^{\infty} h(\tau) R(\tau-\alpha)\, d\tau \simeq h(\alpha) \int_{-\infty}^{\infty} R(t-\alpha)\, d\tau$$

Then:

$$E[y^2(t)] \simeq qE$$

where:

$$q = \int_{-a}^{a} R(\tau)\,d\tau$$

& $E = \int_{-\infty}^{\infty} |h(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\,d\omega$

Proof: Recall:

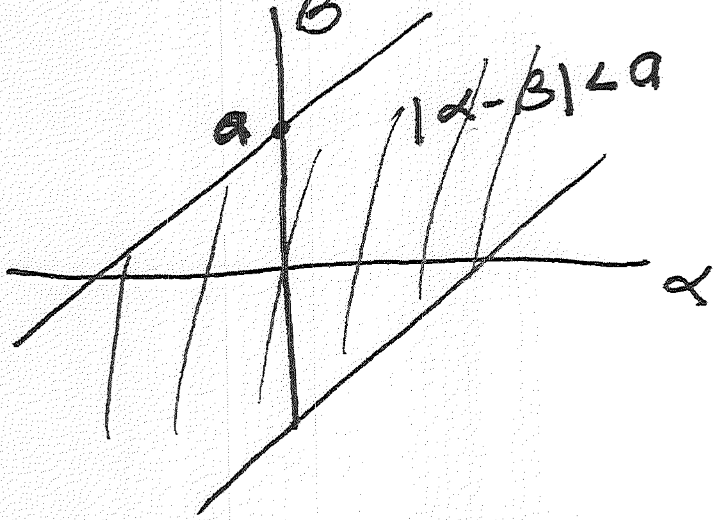
$$R_{YY}(t_1,t_2) = \iint_{-\infty}^{\infty} R_X(t_1-\alpha, t_2-\beta)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta$$

Thus:

$$E[Y^2] = R_{YY}(t,t) = \iint_{-\infty}^{\infty} R_X(t-\alpha, t_2-\beta)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta$$

$$= \iint_{-\infty}^{\infty} R_X(t-\alpha \cdot t+\beta)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta$$

$$= \iint_{-\infty}^{\infty} R_X(\beta-\alpha)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta$$



on this strip, $h(\alpha) \simeq h(\beta)$ by assumption (since α & β are "close")

Thus

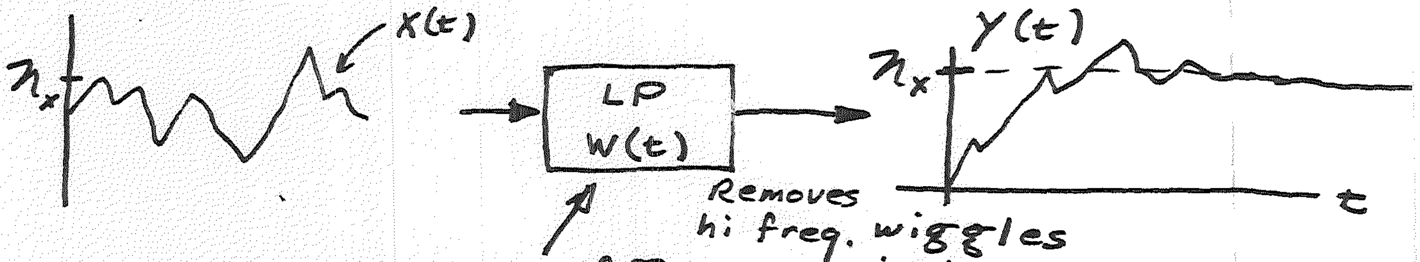
$$E[\underline{Y}^2(t)] = \int_{\alpha=-\infty}^{\infty} h^2(\alpha) \int_{-\infty}^{\infty} R(\beta - \alpha)\, d\beta\, d\alpha$$

$$\simeq \int_{\alpha=-\infty}^{\infty} h^2(\alpha)\; q\; d\alpha$$

$$\simeq E q$$

Analog Techniques

Estimate mean of process, $x(t)$.



$$W(\omega) = \int_{-\infty}^{\infty} w(t)\, e^{-j\omega t}\, dt$$

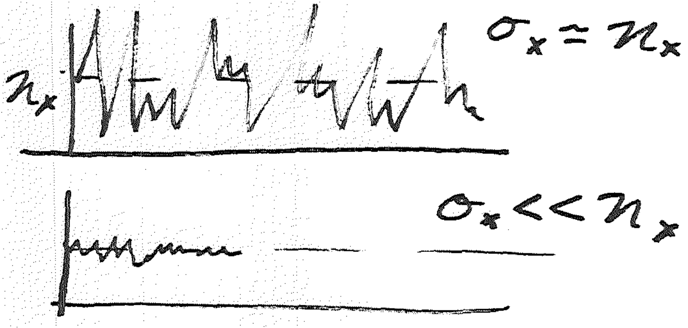
Output:

$$y(t) = \int_{-\infty}^{\infty} x(t-\alpha)\, w(\alpha)\, d\alpha$$

Q: What is $W(\omega)$ such that $y(t) \simeq \eta_x$?

A: ① $\eta_y \simeq \eta_x$

② $\sigma_y << \eta_x$

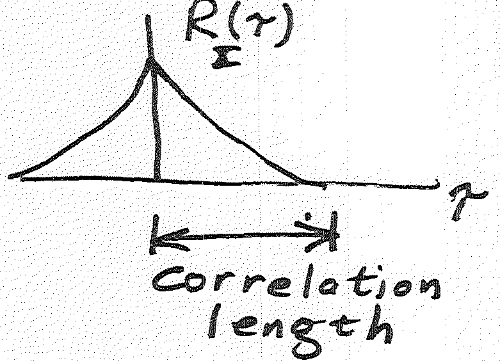


① Set $W(0) = 1$ since

$$\eta_y = W(0)\, \eta_x$$

② Must evaluate σ_y →

Assume bandwidth, ω_c, of LPF is "small" ~~to~~ ~~w.r.t.~~ so that $w(t)$ is "constant" w.r.t. correlation length of $\underline{X}(t)$



Then, recall:

$$E[\underline{Y}^2] \simeq qE \qquad q = \int_{-\infty}^{\infty} R_{\underline{X}}(\tau)\,d\tau$$

Thus

$$\text{var } \underline{Y} \simeq \overset{E[\underline{Y}^2]}{\overset{\shortparallel}{qE}} - \eta_y^2$$

$$= \hat{q}E \qquad ; \hat{q} = \int_{-\infty}^{\infty} C_{\underline{X}}(\tau)\,d\tau$$

(Replace $\underline{Y}$ by $\underline{Y} - \eta_y$)

$$E = \int_{-\infty}^{\infty} w^2(t)\,dt$$

Then, second condition becomes:

$$\hat{q}E \ll \eta_x^2$$

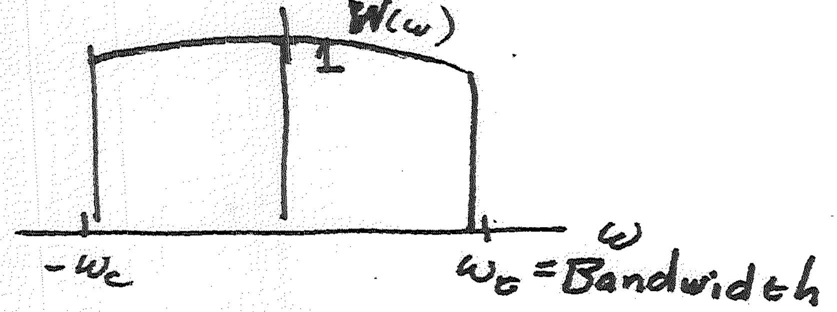
~~Thus, in summary:~~

~~①~~

By Parseval's Theorem:

$$E = \int_{-\infty}^{\infty} w^2(t)\,dt = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} |W(\omega)|^2\,d\omega$$

Assume



$$|W(\omega)| \le W(0) = 1$$

$$\Rightarrow E \le \frac{1}{2\pi}\, 2\omega_c = \frac{\omega_c}{\pi}$$

Thus condition is: $\hat{q}E \le \frac{\hat{q}\omega_c}{\pi} << n_x^2$

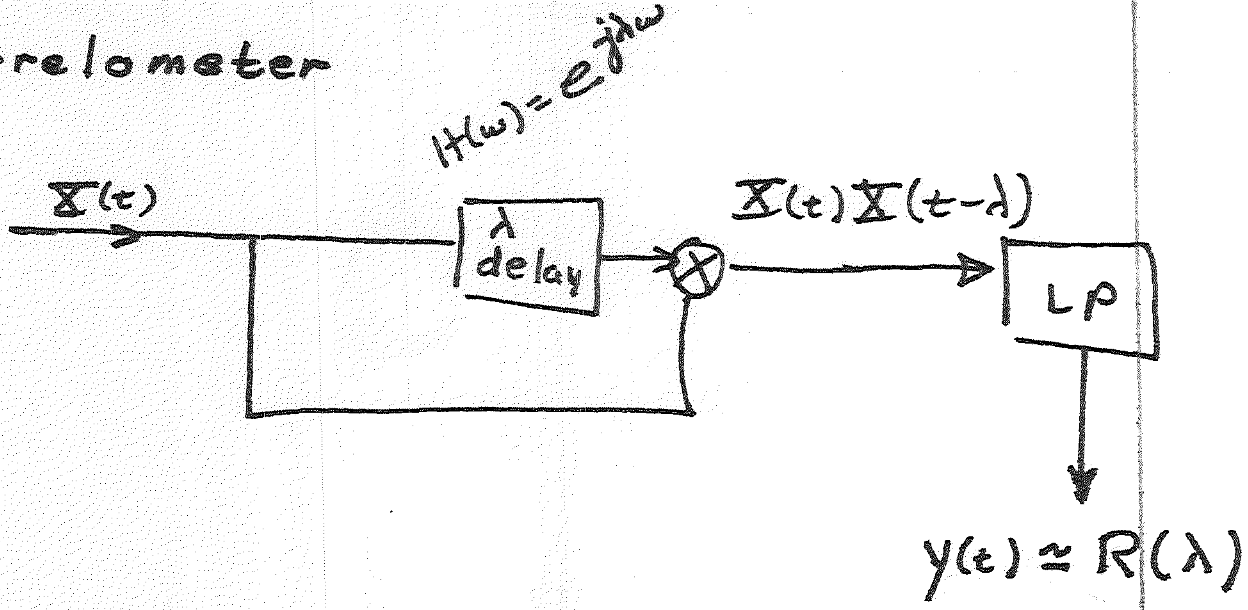
or $\omega_c << \pi n_x^2 / \hat{q}$

Thus, is summary, wish:

① $W(0) = 1$

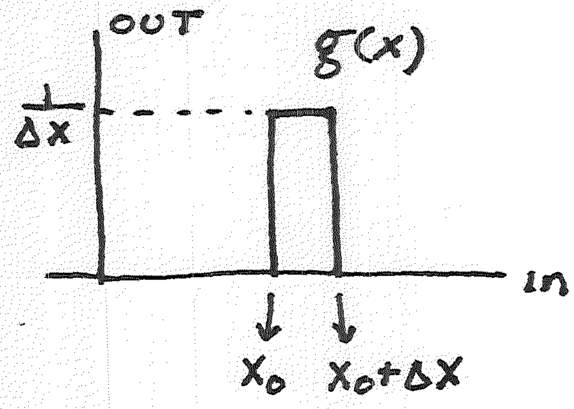
② $\omega_c << \pi n_x^2 / \hat{q}$

Correlometer



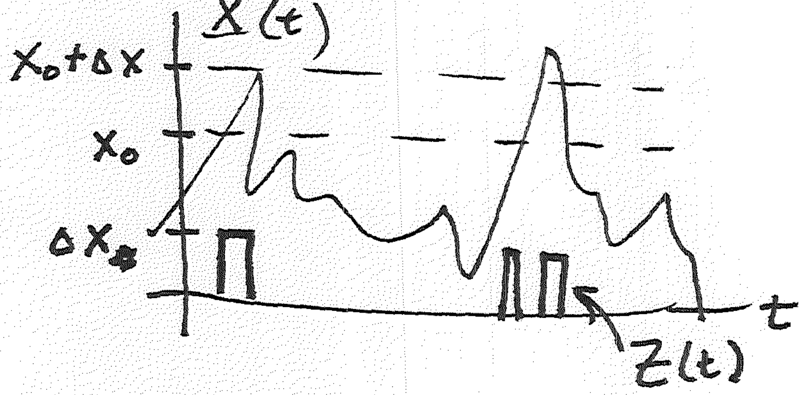
Finding Density Function from Single Sample

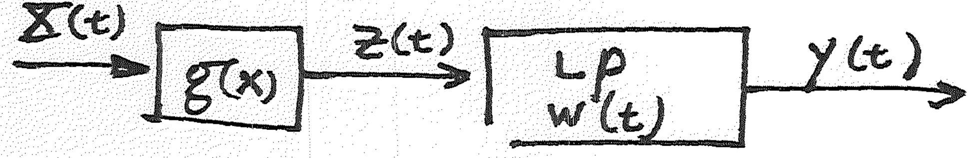
Use ZNL



Then:

$$Z(t) = \begin{cases} \frac{1}{\Delta x} & ; x_0 < X(t) < x_0 + \Delta x \\ 0 & ; \text{otherwise} \end{cases}$$





$$E[Z(t)] \doteq f(x_0)$$

= average time $X(t)$ is 'twixt x_0 & $x_0 + \Delta x$

Must use # stages for entire process.

Solution

EE505, Nov 27, 1996
Exam

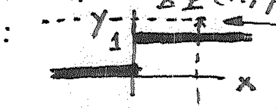
1. $\Phi_X(\omega) = \frac{1}{1+(\frac{\omega}{a})^2} \Rightarrow \Psi_X(\omega) = -\ln\left(1+(\frac{\omega}{a})^2\right)$

$\Psi'(\omega) = -\frac{2\omega/a^2}{1+(\frac{\omega}{a})^2} \Rightarrow \overline{X} = 0$

$\Psi''(\omega) = -\frac{(1+(\frac{\omega}{a})^2)(\frac{2}{a^2}) - (\frac{2\omega}{a^2})^2}{[1+(\omega/a)^2]^2} \Rightarrow \Psi''(0) = \frac{-2}{a^2} = -\text{var } X \Rightarrow \text{var } X = \frac{2}{a^2}$

2. $Y = U(X)$. Find $F_{XY}(x,y)$ for $x>0$ & $y>1$

Line mass:



$x>0$ & $y>1$

$F_{XY}(x,y) = \Pr[X \le x, Y \le y]$

$= \Pr[X \le x, Y \le 1]$; $x>0, y>1$

$= \Pr[X \le x]$ (since $Y \le 1$ is a certainty)

$= F_X(x)$

3. $X \sim e^{-x}U(x)$; $Y \sim y e^{-\frac{y^2}{2}} U(y)$

Rayleigh

$Y = g(X)$.

Transformation for increasing $g(x)$:

$f_Y(y) = \left|\frac{dg^{-1}(y)}{dy}\right| f_X(g^{-1}(y))$

$y e^{-\frac{y^2}{2}} U(y) = \left[\frac{d}{dy} g^{-1}(y)\right] e^{-g^{-1}(y)} \Rightarrow x = \frac{y^2}{2} = g^{-1}(y)$ works $\Rightarrow y = g(x) = \sqrt{2x}$

4. $f_{XY}(x,y) = \gamma y e^{-y(x+1)} U(x)U(y)$

(a) $\gamma = ? \Rightarrow \gamma \int_{y=0}^{\infty}\int_{x=0}^{\infty} y e^{-y(x+1)} dx\,dy = 1 = \gamma \int_{y=0}^{\infty} y e^{-y} \frac{1}{y} e^{-yx}\Big|_0^{\infty} dy = \gamma \int_0^{\infty} e^{-y} dy = \gamma \Rightarrow \gamma = 1$

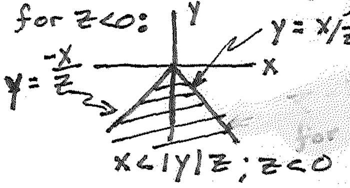
(b) $f_Y(y) = \int_x f_{XY}(x,y)\,dx = \int_{x=0}^{\infty} y e^{-y} \underset{\overset{\wedge}{U(y)}}{e^{-yx}} dx = y \underset{\overset{\wedge}{U(y)}}{e^{-y}} \frac{e^{-yx}}{y}\Big|_0^1 = e^{-y} U(y)$

5. $\Theta_X(\omega) = \ln \Psi_X(\omega)$

$d\Theta_X(\omega)/d\omega = \Psi_X'(\omega)/\Psi_X(\omega) \Rightarrow \Theta_X'(0) = \Psi'(0)/\Psi_X(0) = j\infty \Leftarrow$ UNDEFINED SINCE $\Psi_X(0) = 0$

6. $Z = X/|Y|$; $F_Z(z) = \Pr[Z \le z] = \Pr[X/|Y| \le z] = \Pr[X \le |Y| z]$

for $z<0$:



$= \int_{-x/z}^{x/z} f_{XY}(x,y)\,dx\,dy$; $z<0$

↖ 1-D integral

7. Tchebycheff Inequality (for zero mean):

$\Pr[|X| \ge \varepsilon] \le \sigma^2/\varepsilon^2$

or $\Pr[|X| \le \varepsilon] \ge 1 - \sigma^2/\varepsilon^2$

We want $\Pr[|X| \le T] \ge 0.99 = 1 - \frac{\sigma^2}{T^2} = 1 - \frac{1}{T^2}$ since $\sigma^2 = 1$

$\frac{1}{T^2} = 0.01 \Rightarrow T^2 = 100 \Rightarrow T = 10$

Solutions

EE505 Final Examination

Robert J. Marks II

August 20, 1997; 2:20 to 4:20 PM

The examination is closed book and closed notes. No calculators are allowed. Each student is allowed three sheets of notes. All problems are weighted equally. Work must be done in ink.

All work will be done in a test booklet. No scratch paper is needed.

"And I trust that you will discover that we have not failed the test.", 2 Corinthians 13:6 (English-NIV)

1. Let X and Y be independent random variables and let $Z = X + Y$. Prove or disprove the following general propositions.
   - (a) $\bar{Z} = \bar{X} + \bar{Y}$ $\longrightarrow$ $\overline{Z} = \overline{X+Y} = \overline{X} + \overline{Y} \Leftarrow$ TRUE
   - (b) $\bar{Z^2} = \bar{X^2} + \bar{Y^2}$ $\longrightarrow$ $\overline{Z^2} = \overline{(X+Y)^2} = \overline{X^2} + 2\overline{X}\,\overline{Y} + \overline{Y^2} \neq \overline{X^2} + \overline{Y^2}$ (NOT TRUE IN GENERAL)
   - (c) $\text{var}(Z) = \text{var}(X) + \text{var}(Y)$.
   - (d) $\text{var}(aZ) = a^2 \ \text{var} Z$.

$$\Phi_Z(\omega) = \Phi_X(\omega)\Phi_Y(\omega)$$

$$\Psi_Z(\omega) = \Psi_X(\omega) + \Psi_Y(\omega)$$

$$\Psi_Z''(0) = \Psi_X''(0) + \Psi_Y''(0)$$

Thus var Z = var X + var Y ⇐ True!

$$\text{var}\,aZ = \overline{(aZ)^2} - \overline{aZ}^2 = a^2\left(\overline{Z^2} - \overline{Z}^2\right)$$

$$= a^2 \text{var}\,Z \Leftarrow \text{TRUE}$$

2.

$$Y = \frac{1}{N}\sum_{k=1}^{N} X_k^2$$

where the X_k's are i.i.d. random variables with probability density function

$$f_X(x) = e^{-x}U(x)$$

Estimate the probability density function for the random variable Y when N is large.[1]

Let $Z_k = X_k^2 \Rightarrow \overline{Z_k} = \overline{X_k^2} = 1$

$\overline{Z_k^2} = \overline{X_k^4} = 3! = 6 \Rightarrow \text{var } Z_k = 5$

From problem # 1

$$\overline{\sum_{k=1}^{N} X_k^2} = \sum_{k=1}^{N} \overline{X_k^2} = \sum_{k=1}^{N} 1 = N$$

$$\text{var} \sum_{k=1}^{N} X_k^2 = \text{var} \sum_{k=1}^{N} Z_k = \sum_{k=1}^{N} \text{var } Z_k = \sum_{k=1}^{N} 5 = 5N$$

By Central Limit Theorem:

$$\sum_{k=1}^{N} X_k^2 = \sum_{k=1}^{N} Z_k \sim n(N, \sqrt{5N})$$

and $Y = \frac{1}{N}\sum_{k=1}^{N} X_k^2 \sim n\left(1, \sqrt{\frac{5}{N}}\right)$

or: $$f_Y(y) \approx \frac{1}{\sqrt{2\pi}\sqrt{\frac{5}{N}}} e^{-\frac{(y-1)^2}{5/N}}$$

---

[1] Recall from the last test that the nth moment of each X_k is $(n+1)!$

3. A total of N i.i.d. Bernoulli trials with probability of success p are performed. The outcome of trial m, the random variable X_m, is set to one if there is a success and zero otherwise. We form the sum

$$Y = \sum_{m=1}^{N} X_m.$$

Evaluate the exact probability density function for the random variable Y.

This is simple binomial R.V.

$$p_k = P_r[\underline{Y} = k] = \binom{N}{k} p^k q^{N-k} \; ; \; q = 1-p$$

$$f_{\underline{Y}}(y) = \sum_k p_k \, \delta(y-k)$$

$$= \sum_{k=0}^{N} \binom{N}{k} p^k q^{N-k} \, \delta(y-k)$$

4. The Weibull random variable Y with positive parameters A and B is

$$F_Y(y) = \left[1 - \exp\left(-\frac{y}{A}\right)^B\right] U(y).$$

Let X be a uniform random variable on the interval $(0, 1)$. Given A and B, find a random variable transformation, $Y = g(X)$, to produce a Weibull random variable.

$$Y = g(X)$$

$$y = F_X^{-1}(x) = g(x)$$

OR

$$x = F_X(y)$$

$$= \left[1 - e^{-\left(\frac{y}{A}\right)^B}\right]$$

$$e^{-\left(\frac{y}{A}\right)^B} = 1 - x$$

$$\left(\frac{y}{A}\right)^B = -\ln(1-x)$$

$$\frac{y}{A} = \left[-\ln(1-x)\right]^{1/B}$$

$$y = A\left[-\ln(1-x)\right]^{1/B} = g(x)$$

5. A random variable has a probability density function of

$$f_X(x) = e^{-x}U(x)$$

We take i.i.d. samples from this distribution until we get a number between zero and one - and then stop. Call this last random variable Y. Evaluate the probability density function of Y.

$$F_Y(y) = P_r[Y \le y] = P_r[X \le x \mid 0 \le X \le 1]$$

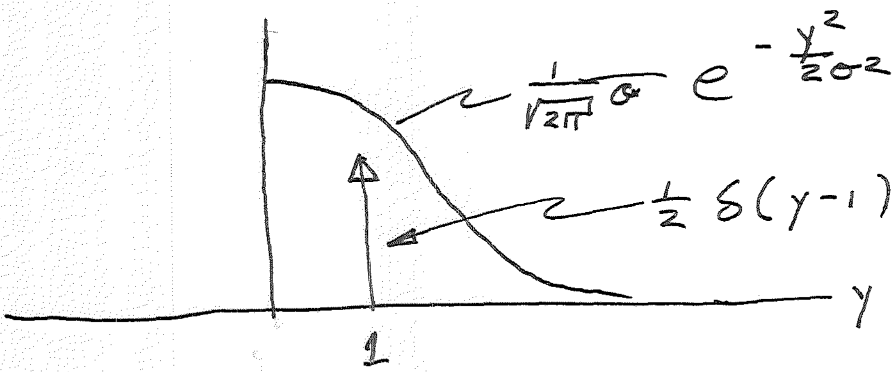
Thus $$f_Y(x) = \begin{cases} A e^{-x} ; & 0 < x < 1 \\ 0 ; & o.w. \end{cases}$$

$$1 = A\int_0^1 e^{-x}dx = A e^{-x}\Big|_1^0 = A(1-e^{-1})$$

$$\Rightarrow A = \frac{1}{1-e^{-1}}$$

$$\therefore f_Y(y) = \begin{cases} \dfrac{e^{-y}}{1-e^{-1}} ; & 0 \le y \le 1 \\ 0 ; & o.w. \end{cases}$$

6. Let X be a zero mean normal random variable with variance σ^2. Let $Y = X$ when X is positive and let $Y = 1$ otherwise. Evaluate and sketch the probability density function for Y.



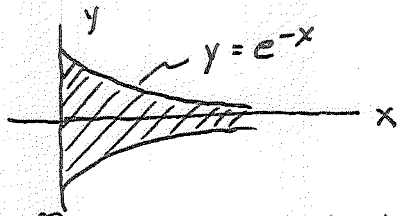
$$f_{\underline{Y}}(y) = \frac{1}{2}\delta(y-1) + \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} U(y)$$

7. A joint probability density function is defined by

$$f_{XY}(x,y) = \begin{cases} A & ;|y| \le e^{-x} \text{ and } y \ge 0 \\ 0 & ;\text{otherwise} \end{cases}$$

(a) Evaluate A.

(b) Evaluate the marginal distribution, $f_Y(y)$.



$f_{XY}(x,y)$

$y = e^{-x}$

$\Rightarrow x = -\ln y$

(a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

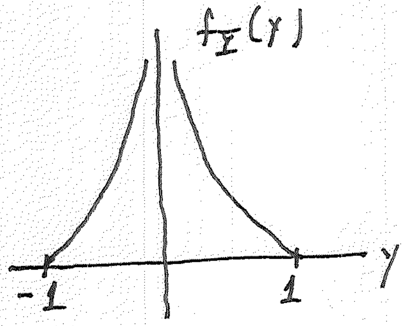
$= A\int_{0}^{\infty}\int_{y=-e^{-x}}^{e^{x}} dx\,dy$

$= A\int_{0}^{\infty} 2e^{-x}\,dx = 2A \Rightarrow A = \frac{1}{2}$

(b) $f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$

$= \frac{1}{2}\int_{x=0}^{-\ln|y|} dx$

$= -\frac{1}{2}\ln|y|$; $|y| < 1$ and 0 o.w.



EE505, EXAMINATION #1 SOLUTIONS

1. d = dogfish twice as hungry

$\Rightarrow$ Effective # of dogfish = 60,000

$$p_d = \frac{6}{6+2+1} = 6/9 = 2/3$$

c = catfish $\Rightarrow p_c = 1/9$, p = perch $\Rightarrow p_p = 2/9$

$$\text{Prob}[k_d = 3, k_c = 0, k_p = 1] = \frac{4!}{3!0!1!}\left(\frac{2}{3}\right)^3\left(\frac{1}{9}\right)^0\left(\frac{2}{9}\right) = 0.263$$

2. $$m_n = E[X^n] = \frac{1}{2c}\int_{-c}^{c} x^n dx = 0 \text{ if } n \text{ is odd}$$

$$= \frac{1}{2c} \cdot 2\int_0^c x^n dx \quad \text{if } n \text{ is even}$$

$$= \frac{1}{c}\,\frac{1}{n+1}\,x^{n+1} = \frac{c^n}{n+1}$$

$$\therefore\ m_n = \begin{cases} 0 & ; \ n \text{ odd} \\ \frac{c^n}{n+1} & ; \ n \text{ even} \end{cases}$$

3. $X \sim N(3,1)$

$$P_r[2 \le X \le 4 \mid X > 2] = \frac{P_r[2 \le X \le 4,\ X > 2]}{P_r[X > 2]}$$

$$= \frac{P_r[2 \le X \le 4]}{P_r[2 \le X \le \infty]}$$

$$= \frac{P_r\left[-1 \le \frac{X-3}{1} \le 1\right]}{P_r\left[-1 \le \frac{X-3}{1} \le \infty\right]}$$

$$= \frac{\text{erf}(1) - \text{erf}(-1)}{\text{erf}(\infty) - \text{erf}(-1)} = \frac{2\,\text{erf}(1)}{\frac{1}{2} + \text{erf}\,1}$$

$$= \frac{2}{\frac{1}{2\,\text{erf}(1)} + 1} \quad ; \ \text{erf}(1) = 0.34134$$

$$\doteq 0.81$$

4 (a) $\Phi(0) = 1 = \exp(e^{+b} - a) \Rightarrow a = e^b$ or $b = \ln a$

(b) $\Psi(s) = \ln \Phi = e^{s+b} - a$

$$\frac{d\Psi}{ds} = e^{b+s} \Rightarrow \Psi'(0) = \eta = e^b = a$$

$$\frac{d^2\Psi}{ds^2} = e^b e^s \Rightarrow \sigma^2 = a = \Psi''(0)$$

5. $E[Y^n] = \frac{1}{2}\int_{-1}^{1} e^{-2x^2 n}\,dx$

$$2nx^2 = \frac{z^2}{2}$$

$$\Rightarrow z = 2\sqrt{n}\,x \Rightarrow dx = \frac{1}{2\sqrt{n}}\,dz$$

$$E[Y^n] = \frac{1}{2}\int_{-2\sqrt{n}}^{2\sqrt{n}} e^{-z^2/2}\,\frac{1}{2\sqrt{n}}\,dz$$

$$= \frac{1}{4\sqrt{n}}\ \sqrt{2\pi}\ 2\ \frac{1}{\sqrt{2\pi}}\int_0^{2\sqrt{n}} e^{-z^2/2}\,dz$$

$$= \frac{1}{2}\sqrt{\frac{2\pi}{n}}\ \mathrm{erf}(2\sqrt{n})$$

$$\Rightarrow E(Y) = \frac{1}{2}\sqrt{2\pi}\ \mathrm{erf}(2) = \frac{1}{2}\sqrt{2\pi}\cdot 0.47726$$

$$= 0.60$$

$$E(Y^2) = \frac{1}{2}\sqrt{\pi}\ \mathrm{erf}(2\sqrt{2}) = \frac{\sqrt{\pi}}{2}\cdot 0.498 = 0.44$$

$$\Rightarrow \mathrm{var}\,Y = 0.084$$

6. $y = g(x) = x^n \Leftarrow$ strictly ↑ for odd n

$$\Rightarrow x = y^{\frac{1}{n}} = g^{-1}(y)$$

$$d\,g^{-1}(y)/dy = \frac{1}{n}\,y^{\frac{1}{n}-1}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{2nc}\,y^{\frac{1}{n}-1} & ;\ |y| \le c^n \\ 0 & ;\text{otherwise} \end{cases}$$

FINAL EXAMINATION

Name. . . . .

Score. . . . .

INSTRUCTIONS

1. Mail your completed examination to:

Dr. Robert J. Marks II
16515 Ashworth Ave. N.
Seattle, WA 98133

Submit your work stapled and in order with this page as the cover sheet. Your envelope must be postmarked no later than June 8, 1987.

2. Since this is a take home exam, neatness and clarity of your presentation are taken into account.

3. Your lowest quiz grade will be dropped. In addition, you may elect to replace a second quiz score with your score on the corresponding problem on this test (i.e. problem 1 for quiz 1, etc.). Any ambiguity or unclarity in this request will result in it being ignored. Make your request, if any, here:

. . . .

4. After completing the exam, please sign the following:

> "I have received no outside (human) help on this examination or, if I have, the names of the people I have consulted are listed below my signiture".

X. . . . .

date. . . . .

PROBLEMS:

1. Four integer numbers are to be encoded in a generalization of a Hamming code. We have the standard table:

1	3	5	7
2	3	6	7
4	5	6	7

The 4 integer numbers are labeled 3,5,6,7. Integer 1, then, is the sum of integers 3,5 and 7. Integer 2 is the sum of 3,6 and 7 and integer 4 is the sum of 5,6 and 7. Thus, the integers 8,10,9,1 would be coded as 19,18,8,20,10,9,1. Suppose, then, 4 other integers were so encoded and sent over a noisy channel. At the receiver, we decode 7,8,1,9,2,5,4. One of these integers is wrong. Find out which one it is and correct it.

2. A bandlimited signal, f(t), has a maximum frequency of B hertz. Assume that we sample <u>in excess</u> of 2B samples per second. The sample taken at the origin, f(0), however, is lost. Show how we can regain this lost sample from those remaining. (Hint: What happens to the replicated spectrum when the sample at the origin is lost?)

3. The signal 8 cos(wt) is sent over a linear time-invariant distortionless channel. The received signal is 4 sin(wt). What is the received signal when the transmitted signal is 4 sin(wt)?

4. A binary string of numbers codes a logic 1 as an isosceles triangle of height A and duration T. A logic zero is the negative of this. Assuming an equal density of ones and zeros, what is the power spectral density of this encoding technique?

5. White gaussian noise has a uniform power spectral density of height N/2. What percentage of the time does the noise waveform exceed one?

6. A transmitted DSB signal undergoes a square law nonlinear transformation. That is, the received modulated signal is the square of what it should be. Is the signal degraded beyond recovery? If not, please explain a process by which it can be regained.

7. Bill the radioman says he can use an envelope detector to demodulate FM. He says you can run the modulated signal through a differentiator and then the envelope detector. Assuming that Bill has an RF differentiator, is he right?

EE505 Examination #1
July 22, 1987
2:20 to 4:30 PM

name_______________________

score_____________/120 pts.

Instructions:

1. Do all of your work in this test booklet.
2. You are allowed one sheet of written notes and a calculator. A table of erf function values is given below.
3. Each problem is worth 20 points.

Table 3-1 $\operatorname{erf} x = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-y^2/2}\, dy = \mathbb{G}(x) - \frac{1}{2}$

x	erf x	x	erf x	x	erf x	x	erf x
0.05	0.01994	0.80	0.28814	1.55	0.43943	2.30	0.48928
0.10	0.03983	0.85	0.30234	1.60	0.44520	2.35	0.49061
0.15	0.05962	0.90	0.31594	1.65	0.45053	2.40	0.49180
0.20	0.07926	0.95	0.32894	1.70	0.45543	2.45	0.49286
0.25	0.09871	1.00	0.34134	1.75	0.45994	2.50	0.49379
0.30	0.11791	1.05	0.35314	1.80	0.46407	2.55	0.49461
0.35	0.13683	1.10	0.36433	1.85	0.46784	2.60	0.49534
0.40	0.15542	1.15	0.37493	1.90	0.47128	2.65	0.49597
0.45	0.17364	1.20	0.38493	1.95	0.47441	2.70	0.49653
0.50	0.19146	1.25	0.39435	2.00	0.47726	2.75	0.49702
0.55	0.20884	1.30	0.40320	2.05	0.47982	2.80	0.49744
0.60	0.22575	1.35	0.41149	2.10	0.48214	2.85	0.49781
0.65	0.24215	1.40	0.41924	2.15	0.48422	2.90	0.49813
0.70	0.25804	1.45	0.42647	2.20	0.48610	2.95	0.49841
0.75	0.27337	1.50	0.43319	2.25	0.48778	3.00	0.49865

Problem 1: In Lake Washington there are 10,000 Catfish, 20,000 Perch and 30,000 Dogfish. The Dogfish are twice as hungry as the Catfish and the Perch. Olie went fishing and caught 4 fish. All the fish were either Catfish, Perch, or Dogfish. After Olie caught each fish, he set it free. What is the probability that three of the fish were Dogfish and one was a Perch? *Express your answer as a single number.*

Problem 2: Compute all of the moments of a random variable that is uniform over the interval of $-c$ to c.

Problem 3: The diameters of apples grown in eastern Washington is modeled as a Gaussian or normal random variable with a mean of three inches and a standard deviation of one inch. A sorting machine rejects those apples whose diameter is less than two inches. After sorting, what is the probability that an apple has a diameter between two and four inches? *Express your final answer as a number*.

Problem 4: A random variable has a moment generating function of exp[exp(s+b)-a] where a is a given parameter.

(a) What is b?

(b) Compute the mean and the variance of this random variable.

Problem 5: The random variable X is uniform on the interval of minus one to one. Let Y = exp(-2X^2). Compute a numerical value for the mean and variance of Y.

Problem 6: Let N be a positive odd integer other than one and let $\mathbf{X}$ denote a random variable that is uniform over the interval of $-c$ to c. Compute the probability density function for $\mathbf{Y} = \mathbf{X}^N$.

mini quiz one:

Baseball player A has a batting average of 0.300 . Batter B's is .200. Manager C rolls a die. If the result is a three or a six, then batter B bats. Otherwise, batter A bats. The batter gets a hit. What is the probability that it was batter A?

mini quiz one:

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Solution $H = HIT$

$$P(H) = P(H/A)P(A) + P(H/B)P(B)$$

$$= (0.3)\left(\frac{2}{3}\right) + (0.2)\frac{1}{3}$$

$$= 0.26667$$

$$P(H,A) = P(H/A)P(A) + P(A/H)P(H)$$

$$\Rightarrow P(A/H) = \frac{P(H/A)P(A)}{P(H)} \Leftarrow \text{Bayes}$$

$$= \frac{(0.300)\,2/3}{0.26667} = 0.75$$

or 75%

mini quiz #2

You receive 3 cards with replacement from a standard deck of 52. Find the probability that:
(a) ...at least two are clubs.
(b) ...at least two are of the same suit.
(c) ...one is an ace and two are kings.

mini quiz #2

You receive 3 cards with replacement from a standard deck of 52. Find the probability that:
(a) ...at least two are clubs.
(b) ...at least two are of the same suit.
(c) ...one is an ace and two are kings.

(a) Three repeated Bernoulli Trials.

$$p = 1/4$$

$$\Pr[\text{2 are clubs } \underline{\text{or}} \text{ three are clubs}]$$

$$= \binom{3}{2} p^2 q + \binom{3}{3} p^3 q^0 = 3(1/4)^2(3/4) + (1/4)^3 = 0.156$$

(b) $=\Pr[\text{at least two are of the same suit}]$

$$= P[\text{at least 2 are clubs}] + P[\text{" " " " diamonds}] + P[\text{" " " " hearts}] + P[\text{" " " " spades}]$$

(The events are mutually exclusive)

$$= 4 \times 0.156 = 0.625$$

(c) Partition: $a_1 = \text{ace}$, $a_2 = \text{king}$; $a_3 = \text{other}$
$p_1 = 1/13$, $p_2 = 1/13$; $p_3 = 11/13$

$$\Pr[k_1 = 1, k_2 = 2, k_3 = 0] = \frac{3!}{1!\,2!\,0!} p_1^1 p_2^2 p_3^0$$

$$= 3(1/13)^3 = 0.0013655$$

FINAL EXAMINATION

INSTRUCTIONS:

1. Mail your exam with this page as a cover sheet to:

Dr. Robert J. Marks II
16515 Ashworth Ave. N.
Seattle, WA 98133

Exams must be postmarked no later than Monday, March 9th, 1987.

2. If you want the graded exam mailed directly to you, include a self addressed stamped envelope. Otherwise, the exam will be returned to Cogswell.
3. Please sign the following:

"I have neither received nor given any information concerning this examination or if I have received or given information, the details of this exchange are given on the back of this page."

X ______________________________
(sign)

______________________ __________
(print your name) (date)

4. Neatness and clarity of the presentation of your results will be taken into account.

EXAMINATION PROBLEMS:

1. A family has three children, none of which are twins or triplets. What is the probability that all three are born on the same day of the year? What is the probability that all three are born on the same day of the year and all three are boys? What is the probability that two of the 3 are boys both born on the same day of the year?
2. Problem 3-5 in Papoulis (p.60).
3. A Poisson process with parameter $\lambda = 2$ occurances per hour is observed for one half of an hour. What is the probability that the number of occurances exceeds two given that the number of occurances exceeds one? Give a single number for your final answer.
4. X and Y are independent random variables. Both are distributed uniformly on the interval from zero to one. Let $Z = XY$.
   (a) Compute $f_Z(z)$.
   (b) Find $Pr(Z \leq 1/2)$.
5. Problem 7-2 in Papoulis (p.170).
6. Problem 8-25 in Papoulis (p.202).
7. Problem 9-1 in Papoulis (p.258)

SOLUTIONS

1. (a) $p = 1/(365.25)^2 = 7.5 \times 10^{-6}$
or 1 chance in 133,407.56

(b) Date of birth is independent of sex. Here

$$p = \frac{1}{(365.25)^2} \cdot \frac{1}{8} = 9.37 \times 10^{-7}$$

or one chance in 1,067,260.50

(c) Children = A, B, C

$$P_r = \text{Prob}[AB\bar{C} + A\bar{B}C + \bar{A}BC + ABC]$$

$$P_r[AB] = \underset{\uparrow}{\frac{1}{4}} \cdot \frac{1}{365.25} = P_r[AC] = P_r[BC]$$

P_r Both Boys

$$\therefore\ P_r = 3\left(\frac{1}{4} \cdot \frac{1}{365.25}\right)\left(\frac{364.25}{365.25}\right) + \frac{1}{8}\,\frac{1}{(365.25)^2}$$

$= 0.00205$ or 1 chance in 488.34

2. $p = 0.6$

$P_r = P_r[550 \le k \le 650] = ?$ if $n = 1000$

$npq = 240 >> 1 \Rightarrow$ use De Moivre-Laplace

$np = 600$

$$P_r = P_r\left[\frac{-50}{\sqrt{240}} \le z = \frac{k-np}{\sqrt{npq}} \le \frac{50}{\sqrt{240}}\right] = 2P_r\left[0 \le z \le \frac{50}{\sqrt{240}}\right]$$

$= 2 \,\text{erf}\, \frac{50}{\sqrt{240}} = 2\,\text{erf}\ 3.23 \Leftarrow$ TO BIG FOR TABLE.

$\therefore$ USE (3-26) on p. 48.

$$\text{erf}\, x = \mathbb{G}(x) - \frac{1}{2} \simeq \frac{1}{2} - \frac{1}{x}\, g(x)$$

$$\therefore\ \text{erf}\, \frac{50}{\sqrt{240}} \simeq \frac{1}{2} - \frac{\sqrt{240}}{50}\, \frac{1}{\sqrt{2\pi}}\, e^{-\left(\frac{50}{\sqrt{240}}\right)^2/2}$$

$= 0.4999323769$

Thus: $P_r = 0.99865$ or 0.999

3. $\lambda = 2$, $T = 1/2 \Rightarrow a = \lambda T = 1 =$ Poisson Parameter

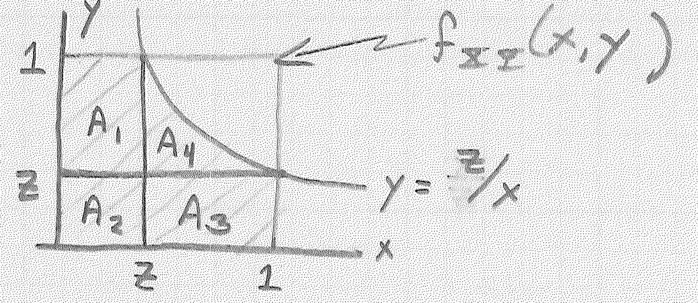
$$P_r = \text{Prob}[X > 2 \mid X > 1] = \frac{P_r[X > 2, X > 1]}{P_r[X > 1]}$$

$$= \frac{P_r[X \ge 3]}{P_r[X \ge 2]} = \frac{1 - P_r[X < 3]}{1 - P_r[X < 2]} = \frac{1 - P_r[X = 0, 1 \text{ or } 2]}{1 - P_r[X = 0 \text{ or } 2]}$$

$$= \frac{1 - e^{-1}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}\right)}{1 - e^{-1}\left(\frac{1}{0!} + \frac{1}{1!}\right)} = 0.304$$

4.(a) $Z = XY$

$F_Z(z) = P_r[XY \le z] = P_r[Y \le z/X] = A_1 + A_2 + A_3 + A_4$



$A_1 + A_2 + A_3 = 2z - z^2$

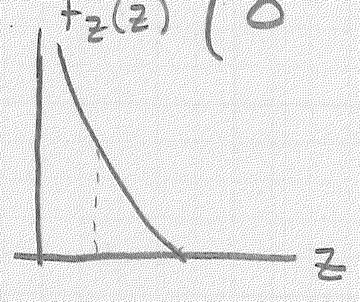
$A_4 = \int_{x=z}^{1} \int_{y=z}^{z/x} dy\,dx = \int_{x=z}^{1} \left[\frac{z}{x} - z\right] dx$

$= z \ln x - zx \Big|_z^1 = -z - (z \ln z - z^2)$

$= z^2 - z - z \ln z$

$\therefore F_Z(z) = z - z \ln z \quad ; \quad 0 \le z \le 1$

$f_Z(z) = \begin{cases} 1 - \left[\ln z + \frac{z}{z}\right] = -\ln z & ; 0 \le z \le 1 \\ 0 & ; \text{otherwise} \end{cases}$



(b) $P_r[Z < \frac{1}{2}] = \int_0^{1/2} \ln z\, dz$

$= F_Z(\frac{1}{2}) - F_Z(0) = \frac{1}{2}(1 - \ln \frac{1}{2})$

$= 0.8466$

5. $E[Z] = E[(X-Y)U(X-Y)]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-y)U(x-y)e^{-x}U(x)e^{-y}U(y)\,dx\,dy$$

$$= \int_{y=0}^{\infty}\left[\int_{x=y}^{\infty}(x-y)e^{-x}\,dx\right]e^{-y}dy = \frac{1}{2}$$

6. $\eta_r = 500$

$\sigma_r^2 = \frac{1}{100}\int_{-50}^{50} r^2\,dr = \frac{50^2}{3}$

$X = r_1 + r_2 + r_3 + r_4$

$\eta_x = 4 \cdot 500 = 2000$

$\sigma_x^2 = 4\,\frac{50^2}{2} = \frac{10^4}{3}$

From CLT,

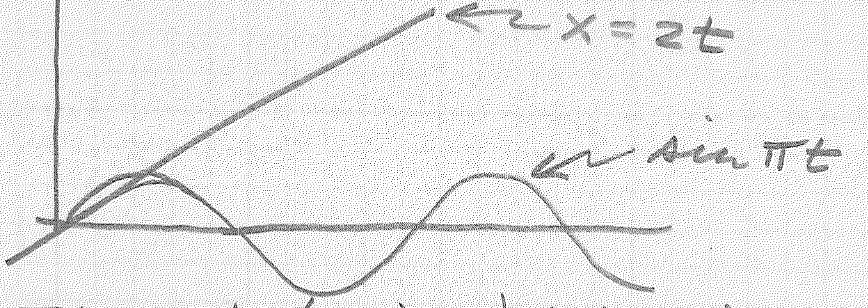
$X \sim N\left(2000;\ \frac{10^2}{\sqrt{3}}\right)$

Thus

$P_r[1900 < X < 2100] = 2\,\mathrm{erf}\,\frac{100}{10^2/\sqrt{3}}$

$= 0.9169$

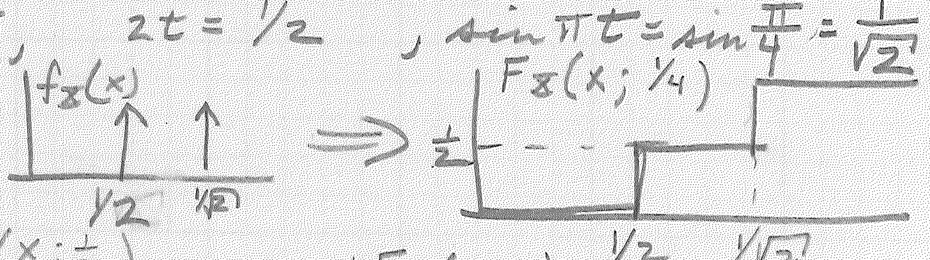
7. $P_r(X(t) = 2t) = \frac{1}{2} = P_r[X(t) = \sin \pi t]$



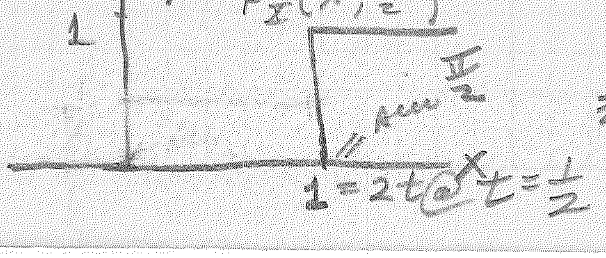
(a) $E\,X(t) = \frac{1}{2}(2t) + \frac{1}{2}(\sin \pi t)$

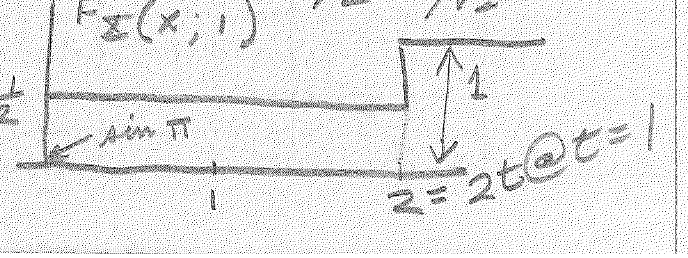
$= (2t)\,P_r[X(t) = 2t] + \sin \pi t\ P_r[X(t) = \sin \pi t]$

(b) @ $t = 0.25$, $2t = 1/2$, $\sin \pi t = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$



similarly:





Final Examination

name ______________________

score ____________

1. Prob 5·3
2. Prob 5·14
3. Prob 8-23
4. Prob 8-24
5. Prob 9-1
6. A discrete stochastic process, $x(n)$, is normalized to $y(n) = A\,x(n)$ before A/(fixed point) D conversion. We would like to have $|y(n)| \leq 1$ to avoid clipping. Assuming that $x(n)$ is zero mean and $\operatorname{var} x(n) = \sigma_x^2$ is known, find A so that

$$\Pr[\,|y(n)| \geq 1\,] \leq 1/16.$$

Instructions:

1. No outside human help.
2. All problems are equally weighted.
3. In take home exams, neatness counts.
4. Mail to:

Dr. Robert Marks
16515 Ashworth Ave. N.
Seattle Wa 98133

postmarked no later than June 9, 1986

5. Use this sheet as a cover. Staple your work together in order.
6. Sign the following:

"All outside references that I have used (human or other) are listed on the back side of this sheet."

X ______________________ ____________

sign date

P1

EE 505
8-20-86
Midterm # 2

Name ______________________

Score ______________

Instructions:

1. Closed notes, closed book.
2. Two sheets of notes (stapled) and a calculator are okay
3. Test time: 2:20 to 4:30 pm sharp.
4. All problems are equally weighted.
5. Do all of your work in this test booklet.
6. After the test is graded, on campus students can pick up their test and grades at the EE main office. (No grades can be given over the phone). TV E students will have their tests returned as usual.

"... of making many books
there is no end; and
much study is a weariness
of the flesh"

Eccl 12:12

-1-

P2

(a) Find $f_Z(z)$ given $f_{XY}(x,y)$ when $Z = XY$.

(b) Apply your results in (a) when

$$f_{XY}(x,y) = \begin{cases} \frac{x}{\sqrt{2\pi}} e^{-(xy)^2/2} & ; 1 \le x \le 2 \\ 0 & ; \text{otherwise} \end{cases}$$

P3

Let $\vec{X}$ denote an N dimensional vector of iid random variables with mean η and variance σ^2. Let $\vec{Y} = \underline{A}\,\vec{X}$ where $\underline{A}$ is some given N by N matrix; ie

$$Y_m = \sum_{n=1}^{N} a_{mn} X_n \quad ; \; 1 \le m \le N$$

For large N, estimate the first order density of Y_m, as well as its mean and expected value.

The R.V.'s, $\{X_n \mid 1 \le n \le 5\}$ are iid and uniform on the interval $(-1, 1)$.

Compute:

$$E\left[(X_1 + X_2)^3 X_3 + (X_4 + X_5^2)^2 \right]$$

p.5

The joint probability density function:

$$f_{XY}(x,y) = 8y^2 e^{-2y} e^{-2xy}\, U(x)\, U(y)$$

has a marginal density:

$$f_Y(y) = 4y e^{-2y}\, U(y)$$

Given that $Y = 1/2$, what is a good estimate of X? [$U(\cdot)$ = unit step]

P6

The stochastic process $\mathbf{X}(t)$ is defined by:

$$\mathbf{X}(t) = 1 + t\, e^{-\alpha t}$$

where the random variable α is uniformly distributed on $(0,1)$. Consider the random variable

$$Z = \int_0^{\mathbf{Y}} \mathbf{X}(t)\,dt$$

Given that $\mathbf{Y}$ is also uniform on $(0,1)$, and that $\mathbf{Y}$ and $\mathbf{X}$ are independent, compute:

$$\eta_z = E[Z]$$

Hector Gleason manufactures fixed frequency oscillators. Two thirds of his units work. When turned on, they respond $\sin(\omega t)\,U(t)$ where ω is the fixed frequency. The other one third fizzles according to $e^{-\alpha t}U(t)$ where α is always the same. Let $X(t)$ be the waveform we obtain from a Gleason oscillator.

(a) What is the first order density,

$$f_{X(t)}(x\,;\,t) = ?$$

(b) What is the correlation function:

$$R_X(t_1, t_2) = ?$$

P8

WORK SHEET

P9

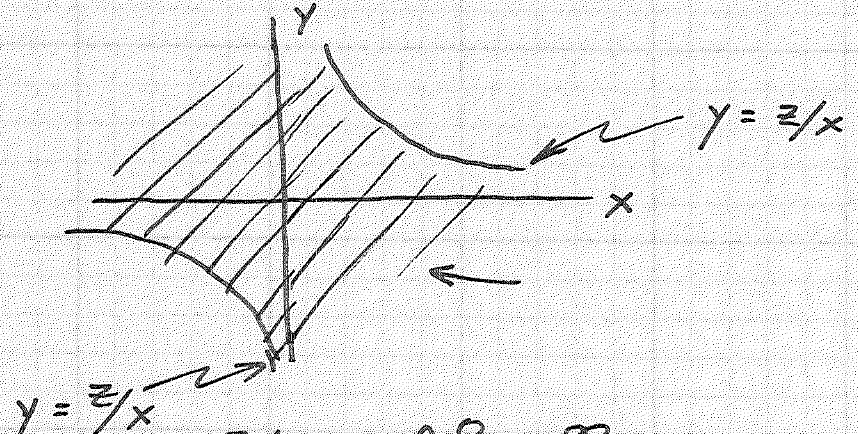
WORK SHEET

EE505, Midterm #2 solutions

1.(a) $\mathbf{Z} = \mathbf{X}\mathbf{Y}$

$$F_{\mathbf{Z}}(z) = P_r[\mathbf{Z} \le z] = P_r[\mathbf{X}\mathbf{Y} \le z]$$

$$= P_r[\mathbf{Y} \le z/\mathbf{X},\ \mathbf{X} \ge 0 \ \underline{\text{or}}\ \mathbf{Y} \ge z/\mathbf{X},\ \mathbf{X} < 0]$$



$$F_{\mathbf{Z}}(z) = \left[\int_{x=0}^{\infty}\int_{y=-\infty}^{z/x} + \int_{x=-\infty}^{0}\int_{y=z/x}^{\infty}\right] f_{\mathbf{X}\mathbf{Y}}(x,y)\,dx\,dy$$

$$f_{\mathbf{Z}}(z) = \int_{x=0}^{\infty} \frac{1}{x} f_{\mathbf{X}\mathbf{Y}}\left(x, \frac{z}{x}\right) dx + \int_{x=-\infty}^{0} \left(-\frac{1}{x}\right) f_{\mathbf{X}\mathbf{Y}}\left(x, \frac{z}{x}\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{|x|} f_{\mathbf{X}\mathbf{Y}}\left(x, \frac{z}{x}\right) dx$$

(b) $$f_{\mathbf{X}\mathbf{Y}}(x,y) = \frac{x}{\sqrt{2\pi}} e^{-\frac{(xy)^2}{2}} \ ; \ 1 \le x \le 2$$

$$f_{\mathbf{Z}}(z) = \int_1^2 \frac{1}{x} \frac{x}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = N(0,1)$$

2. A central limit theorem problem.

$$\eta_m = E\,Y_m = \sum_{n=1}^{N} a_{mn}\,\eta_n$$

$$= \eta \sum_{n=1}^{N} a_{mn}$$

$$\sigma_m^2 = \operatorname{var} Y_m = \sum_{n=1}^{N} \operatorname{var}(a_{mn} X_n)^2$$

$$= \sigma^2 \sum_{n=1}^{N} a_{mn}^2$$

Y_m, in general, will be approxamately normal:

$$f_{Y_m}(y) \sim N(\eta_m, \sigma_m)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_m}\, e^{-\frac{(y-\eta_m)^2}{2\sigma_m^2}}$$

$$e = E\left[(X_1+X_2)^3 X_3 + (X_4 + X_5^2)^2\right]$$

$$= E\left[(X_1+X_2)^3\right] \cancel{E[X_3]}^{\,0}$$

$$+ E[X_4^2] + 2\cancel{E[X_4]}\,E[X_5^2] + E\,X_5^4$$

$$= E[X_4^2] \overset{0}{+} E[X_5^4]$$

$$E\,X^{2n} = \frac{1}{2}\int_{-1}^{1} x^{2n}\,dx = \frac{x^{2n+1}}{2\cdot(2n+1)}\Big|_{-1}^{1} = \frac{1}{2n+1}$$

Thus $e = \frac{1}{3} + \frac{1}{5} = 8/15$

Solution

4. Use minimum MSE $= \hat{X} = E[X/Y]$

$$f_X(x/y) = f_{XY}(x,y)/f_Y(y)$$

$$= 2y\, e^{-2xy}\, U(x)\,U(y)$$

$$E[X/Y] = \int_{-\infty}^{\infty} x\, f_X(x/y)\,dx$$

$$= 2y \int_0^{\infty} x\, e^{-2xy}\,dx\; U(y)$$

Integrate by parts:

$$u = x \qquad dv = e^{-2xy}dx$$

$$du = dx \qquad V = \frac{-1}{2y} e^{-2xy} dx$$

$$\Rightarrow E[X/Y] = 2y\left[\underbrace{\cancel{\frac{-x}{2y} e^{-2xy}\Big|_0^{\infty}}}_{0} + \frac{1}{2y}\int_0^{\infty} e^{-2xy}\,dx\right] \times U(y)$$

$$= 2y\left[\frac{1}{2y}\cdot\frac{1}{2y}\right] = 1/2y$$

$$Y = \tfrac{1}{2} \Longrightarrow \hat{X} = 1 \text{ is minimum mse}$$

5. $$E\,Z = E\int_0^{\mathbf{Y}} \mathbf{X}(t)\,dt$$
$$= E\left[E\int_0^{y} \mathbf{X}(t)\,dt \mid \mathbf{Y}=y\right]$$
$$= E\left[\int_0^{y} E[\mathbf{X}(t)]\,dt \mid \mathbf{Y}=y\right]$$
$$E\,\mathbf{X}(t) = E\,[1+t\,e^{-\alpha t}]$$
$$= 1 + t\int_0^1 e^{-\alpha t}\,d\alpha = 1 - e^{-\alpha t}\Big|_0^1$$
$$= 1 + [1 - e^{-t}] = e^{-t}$$

Thus:
$$E\,z = E\int_0^{y} e^{-t}\,dt \mid \mathbf{Y}=y$$
$$= E\;1 - e^{-\mathbf{Y}}$$
$$= 1 - \int_0^1 e^{-y}\,dy$$
$$= 1 - [1 - e^{-1}] = e^{-1} = \eta_z$$

6.(a) $$f_{\mathbf{X}(t)}(x;t) = \frac{2}{3}\,\delta(x - \sin\omega t\,U(t)) + \frac{1}{3}\,\delta(x - e^{-\alpha t}U(t))$$

(b) $$R_{\mathbf{X}}(t_1,t_2) = E[\mathbf{X}(t_1)\mathbf{X}(t_2)]$$
$$= \int_{-\infty}^{\infty}\!\!\int f_{12}(x_1,x_2)\;x_1\,x_2\,dx_1dx_2$$

where $x(t_1)=x_1$, $x(t_2)=x_2$ and
$$f_{12}(x_1,x_2) = f_{\mathbf{X}(t_1)\mathbf{X}(t_2)}(x_1,x_2;t_1,t_2)$$

Now:
$$f_{12}(x_1,x_2) = f_{12}(x_1,x_2 \mid \mathbf{X}=\sin\omega t\,U(t))\,P_r[\mathbf{X}=\sin\omega t\,U(t)] + f_{12}(x_1,x_2 \mid \mathbf{X}=e^{-\alpha t}U(t))\,P_r[\mathbf{X}=e^{-\alpha t}U(t)]$$
$$= \frac{2}{3}\,\delta(x_1 - \sin\omega t_1\,U(t_1))\,\delta(x_2 - \sin\omega t_2\,U(t_2)) + \frac{1}{3}\,\delta(x_1 - e^{-\alpha t_1}U(t_1))\,\delta(x_1 - e^{-\alpha t_2}U(t_2))$$

and:
$$R(t_1,t_2) = \left[\frac{2}{3}\sin(\omega t_1)\sin(\omega t_2) + \frac{1}{3}e^{-\alpha(t_1+t_2)}\right]U(t_1)\,U(t_2)$$

EE505 EXAMINATION

name________________

score ____ / 150

★ Instruction:

1. Test time : 2:20 to 4:20 PM
   Mon, July 28
2. Closed Book & Notes.
   One Sheet of Notes and Calculator OK.
3. Do all your work in the test booklet.

★ Information:

- All problems are worth 25 pts.

★ Hints:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \cosh x + \sinh x$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \; ; \; \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\operatorname{sech} x = 1/\cosh x \; ; \; \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx} \operatorname{sech} x = - \operatorname{sech} x \; \tanh x$$

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$$

$$2\cosh x = e^x + e^{-x}$$

$$2 \sinh x = e^x - e^{-x}$$

$$\operatorname{sech}(0) = \cosh(0) = 1$$

$$\sinh(0) = \tanh(0) = 0$$

$$\int_{-\infty}^{\infty} b \operatorname{sech}(\pi b x)\, e^{j\omega x}\, dx = \operatorname{sech}(\omega/2b)$$

- 1 -

A fair coin ($p = q = 1/2$) is flipped 1000 times. Compute the probability that 500 were heads.

X is a Poisson random variable with parameter a. What is the probability that X is even? Is it greater than $1/2$?

Bill eats only chili-dogs and olive pizzas. Chili-dogs give him heartburn 10% of the time. The olive pizzas are worse. They give him heartburn 20% of the time. Bill eats twice as many chili dogs as pizzas. Bill has heartburn. What is the probability it was caused by a chili-dog?

The random variable $\mathbf{X}$ has a pdf of $f_{\mathbf{X}}(x) = A \operatorname{sech}(\pi x/a)$ where "a" is a specified parameter

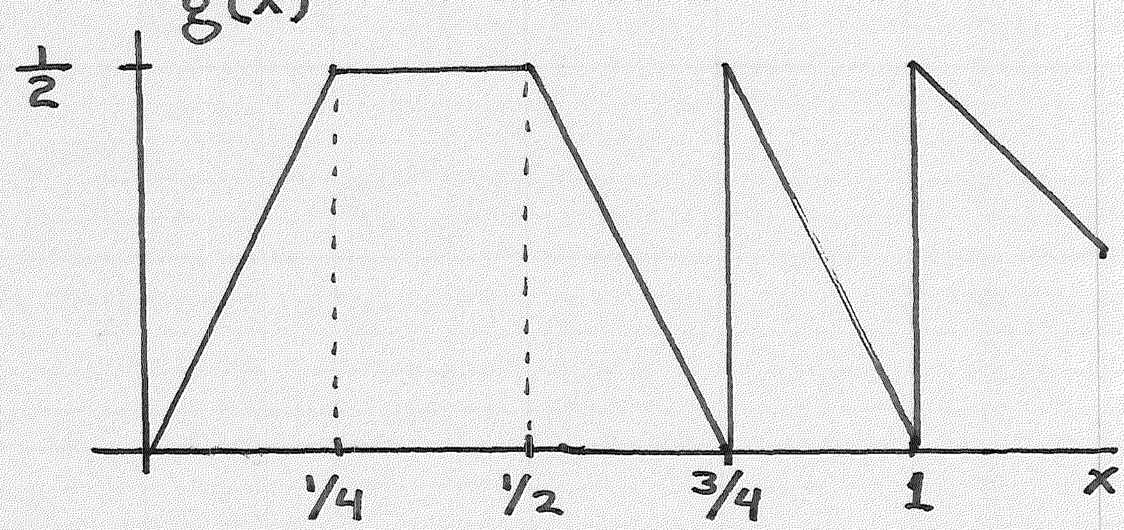
(a) Find A.

(b) Compute $E[\mathbf{X}]$ and var $\mathbf{X}$.

(Do not ask proctor for $\int_{-\infty}^{\infty} x \operatorname{sech} x \, dx$ or $\int_{-\infty}^{\infty} x^2 \operatorname{sech} x \, dx$. Rather, see the "Hints")

-6-

A random variable, $\mathbf{X}$, is uniform on the interval $(0, 1)$. Let $\mathbf{Y} = g(\mathbf{X})$:



Sketch $f_{\mathbf{Y}}(y)$.

The random variable, $\underline{X}$, is the total time a lightbulb is functional given that it was turned on at time = 0. Assume

$$f_{\underline{X}}(x) = \alpha e^{-\alpha x} \; U(x)$$

Suppose that the bulb worked to time t. Find $f_{\underline{X}}(x \mid \underline{X} > t)$ and note the lightbulb is as good as new.

EE505 Final
Part 1
Tues, 8-14-84
noon to 1 P.M.

Name ____________________
Grade ________ /100

1. Casey, the baseball player, has a batting average of 0.300 (ie, $p \approx 0.3$ in a Bernoulli trial). Estimate the probability he gets over 850 hits (successes) in his next 3000 at bats (trials).

2

Let $A = \frac{1}{N} \sum_{n=1}^{N} \mathfrak{X}_n$ where the $\mathfrak{X}_n$'s are iid:

$$f_{\mathfrak{X}_n}(x) = \frac{\alpha/\pi}{\alpha^2 + x^2} \quad ; n = 1, 2, \dots, N$$

Recall $\Phi_{\mathfrak{X}_n}(\omega) = \exp[-\alpha\,|\omega|]$. Compute $f_A(x)$

iid = independent & identically distributed

4

Define $A = \frac{1}{N} \sum_{n=1}^{N} X_n$ where the X_n's are iid with unit variance and zero mean. Compute a lower bound on the probability that A lies between $-a$ and a where $a > 0$ is specified and N is <u>not</u> large enough to apply the central limit theorem.

EE505

Solutions

4. Casey, the baseball player, has a batting average of 0.300 (ie, $p \simeq 0.3$ in a Bernoulli trial). Estimate the probability he gets over 850 hits (successes) in his next 3000 at bats (trials).

Solution

Can use central limit theorem:

$n = 3000$, $p = 0.3$, $q = 0.7$

$\eta = np = 900$, $\sigma^2 = npq = 630$

$$\Pr[k \leq 850] \simeq G\left[\frac{850-900}{\sqrt{630}}\right]$$

$$= G[-1.99] = 0.0233 \Leftarrow \text{From table}$$

$$\therefore \Pr[k > 850] = 0.9767 \simeq 98\%$$

2

Let $A = \frac{1}{N} \sum_{n=1}^{N} \mathbf{X}_n$ where the $\mathbf{X}_n$'s are iid:

$$f_{\mathbf{X}_n}(x) = \frac{\alpha/\pi}{\alpha^2 + x^2} \quad ; n = 1, 2, \ldots, N$$

Recall $\Phi_{\mathbf{X}_n}(\omega) = \exp[-\alpha\,|\omega|]$. Compute $f_A(x)$

<u>Solution</u>

$$\begin{aligned}
\Phi_A(\omega) &= E[e^{j\omega A}] \\
&= E\left[e^{j\omega \frac{1}{N} \sum_{n=1}^{N} \mathbf{X}_n}\right] \\
&= \prod_{n=1}^{N} E\left[e^{j\omega \mathbf{X}_n / N}\right] \quad ; \text{since iid} \\
&= \prod_{n=1}^{N} e^{-\alpha|\omega/N|} \\
&= \left[e^{-\alpha|\omega/N|}\right]^N = e^{-\alpha|\omega|}
\end{aligned}$$

$$\Longrightarrow f_A(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}$$

Same as $f_{\mathbf{X}}(x)$!

(no central limit theorem here!)

4

Define $A = \frac{1}{N} \sum_{n=1}^{N} X_n$ where the X_n's are iid with unit variance and zero mean. Compute a lower bound on the probability that A lies between $-a$ and a where $a > 0$ is specified and N is <u>not</u> large enough to apply the central limit theorem.

Solution

$E[A] = 0$, $\text{var}\, A = \frac{1}{N} = \sigma_A^2$

Use Chebychev inequality:

$$P_r[|A| < k\sigma_A] \geq 1 - \frac{1}{k^2}$$

Set $k\sigma = a$

$$P_r[|A| < a] \geq 1 - \left(\frac{\sigma_A}{a}\right)^2$$

$$= 1 - \frac{1}{Na^2}$$

Note:

$$P_r[|A| < \epsilon] \xrightarrow[N \to \infty]{} 1 \quad \text{for all } \epsilon$$

(A Law of large numbers)

EE505 Final
Part II
Thurs, 8-16-84
noon - 1 p.m.

Name____________________

Score____________ /100

1. Let Υ be a gaussian random variable with mean η and variance σ^2. Define the stochastic process

$$X(t) = \Upsilon \quad \text{for all } t.$$

Compute:

(a) $\eta_x(t)$

(b) $R_x(t_1, t_2)$

(c) $C_x(t_1, t_2)$

(d) var $X(t)$

(e) Is $X(t)$ wss?

2

$X(t)$ is a WSS stochastic process with a first order density:

$$f_X(x) = e^{-x}\,U(x)$$

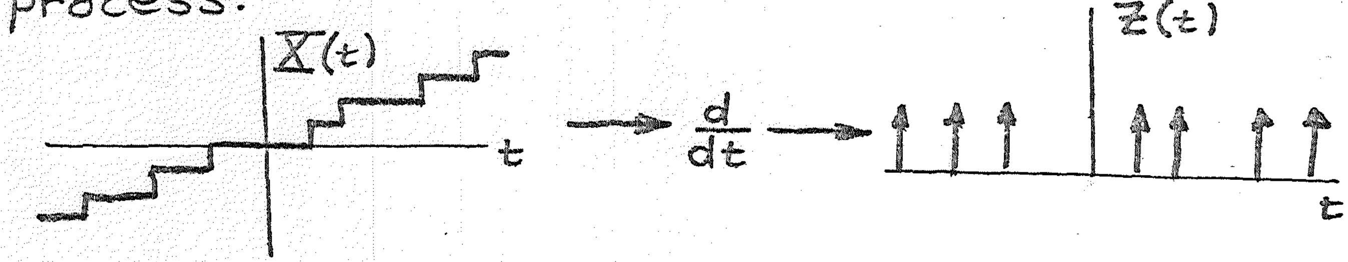
and autocorrelation:

$$R_X(\tau) = e^{-|\tau|}$$

What percentage of the time will $X(t)$ exceed 1?

NO 3

Recall the differentiation of the Poisson process:



we showed that $E[Z(t)]=\lambda$ and that

$$R_Z(\tau)=\lambda^2+\lambda\,\delta(\tau)$$

Is $Z(t)$ mean ergodic? Show your work.

NO 4

Let $x(t)$ be stationary white noise:

$$R_x(\tau) = q\,\delta(\tau) \;,\; \eta_x = 0$$

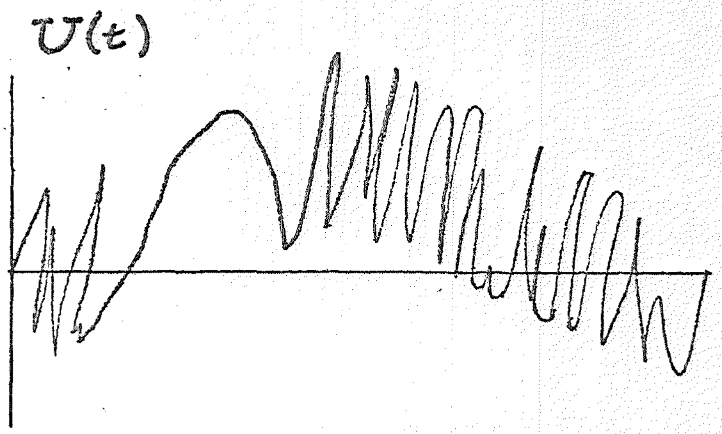
Let:

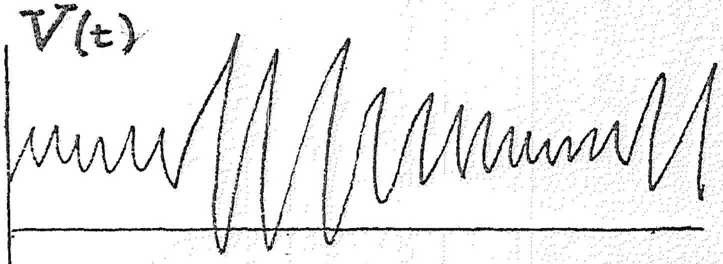
$$\mathbb{X}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

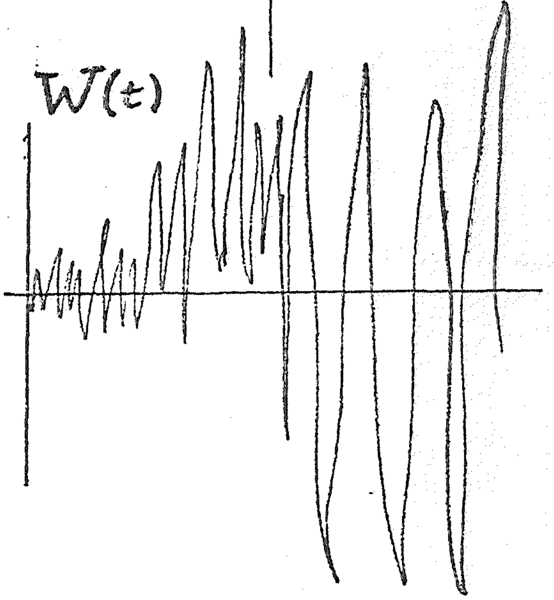
(a) Compute the autocorrelation of $\mathbb{X}(\omega)$.

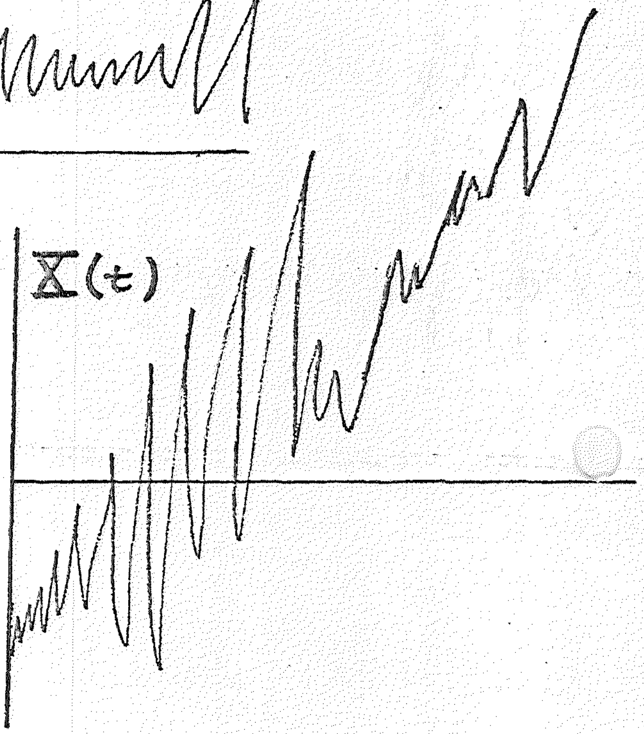
(b) Is $\mathbb{X}(\omega)$ WSS (widesense stationary)?

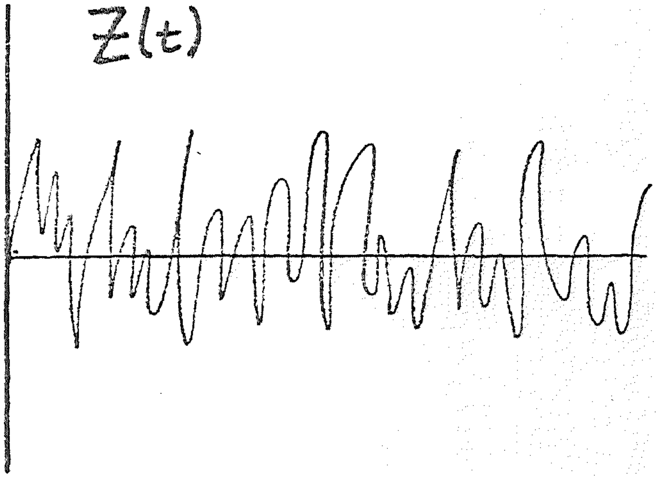
5











Fill in the blank. with the <u>best</u> waveform. Use an answer only once. No guessing penalty

(a) increasing mean ____________

(b) increasing variance ____________

(c) WSS

(d) constant mean ____________ ____________

(e) constant variance ____________

Solutions EE505

1. Let $\underline{Y}$ be a gaussian random variable with mean η and variance σ^2. Define the stochastic process

$$\underline{X}(t) = \underline{Y} \quad \text{for all } t.$$

Compute:

(a) $\eta_x(t)$

(b) $R_x(t_1, t_2)$

(c) $C_x(t_1, t_2)$

(d) var $\underline{X}(t)$

(e) Is $\underline{X}(t)$ WSS?

Solution

(a) $\eta_x = E\underline{X} = E\underline{Y} = \eta$

(b) $R_x(t_1, t_2) = E[\underline{Y}^2] = \sigma^2 + \eta^2$

(c) $C_x(t_1, t_2) = R_x(t_1, t_2) - \eta^2 = \sigma^2$

(d) var $\underline{X} = C_x(t, t) = \sigma^2$

(e) Yes!

2

$X(t)$ is a WSS stochastic process with a first order density:

$$f_X(x) = e^{-x} U(x)$$

and autocorrelation:

$$R_X(\tau) = e^{-|\tau|}$$

What percentage of the time will $X(t)$ exceed 1?

Solution

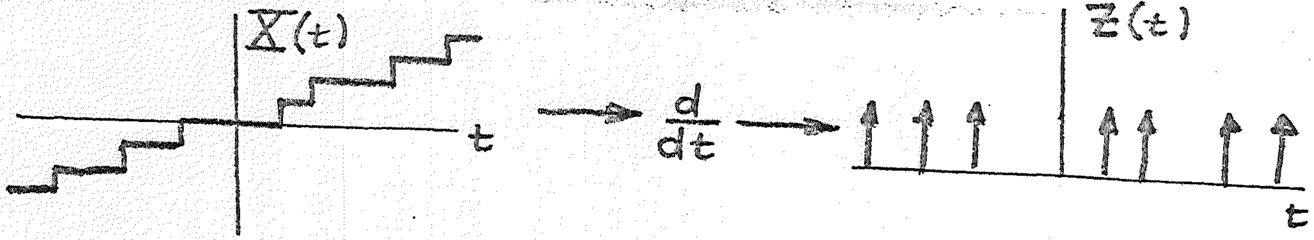
$$Pr[X(t) > 1] = \int_1^\infty e^{-x} dx$$

$$= e^{-x} \Big|_\infty^1 = e^{-1} = 36.8\%$$

(independent of R_X)

3 No

Recall the differentiation of the Poisson process:



we showed that $E[Z(t)] = \lambda$ and that

$$R_Z(\tau) = \lambda^2 + \lambda\,\delta(\tau)$$

Is $Z(t)$ mean ergodic? Show your work.

Solution A sufficient condition for mean ergodicity is $Z(t)$ is wss and $\int_{-\infty}^{\infty} |C_Z(\tau)|\,d\tau < \infty$

Since
$$C_Z(\tau) = R_Z(\tau) - \eta_Z^2$$
$$= \lambda\,\delta(\tau),$$

$$\int_{-\infty}^{\infty} |C_Z(\tau)|\,d\tau = \lambda < \infty$$

yes, Z is mean ergodic

4

Let $x(t)$ be stationary white noise:

$$R_x(\tau) = q\,\delta(\tau) \quad , \eta_x = 0$$

Let:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

(a) Compute the autocorrelation of $X(\omega)$.

(b) Is $X(\omega)$ WSS (wide sense stationary)?

Solution (a) If $x(t)$ is WSS, then $X(\omega)$ is white with

$$R_X(u, v) = 2\pi\, S_x(u)\, \delta(u - v)$$

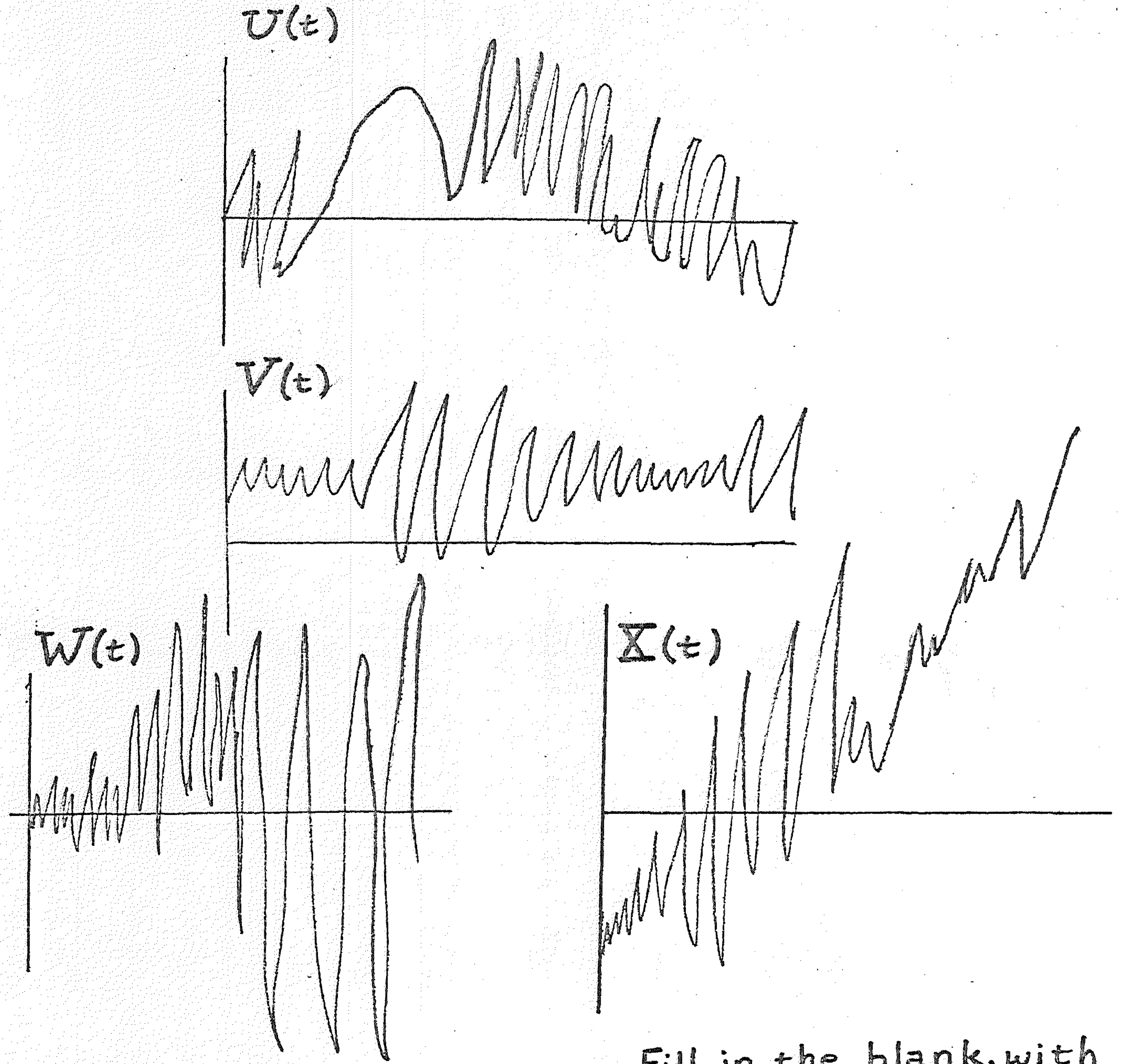
$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j\omega\tau}\, d\tau$$

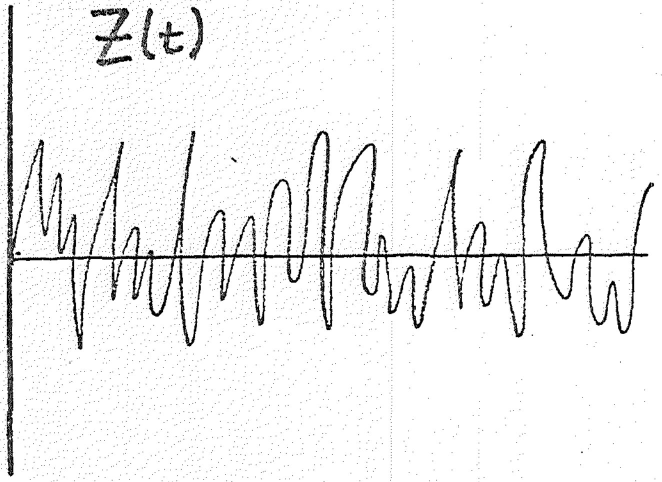
$$= q$$

$$\Rightarrow R_X(u, v) = 2\pi q\, \delta(u - v)$$

(b) Yes. $R_X(u) = 2\pi q\, \delta(u)$

5





Fill in the blank. with the <u>best</u> waveform. Use an answer only once. No guessing penalty

(a) increasing mean ____ X(t)
(b) increasing variance ____ W(t)
(c) WSS ____ Z(t)
(d) constant mean ____ V(t)
(e) constant variance ____ U(t)

DEPARTMENT OF ELECTRICAL ENGINEERING
University of Washington

EE505
Midterm
7/24/84

Name________________________

Score________ / 100

Instructions:
1. Each question has equal weight.
2. The test is closed book and notes. You are allowed one page of notes.
3. Please do all of your work in this test booklet.
4. The test will begin at noon and stop promptly at 1:30.

Problem 1: Multiple Choice (No guessing penalty)

A density function is equal to Ax^2 for $0 < x < 1$ and is otherwise zero.

Answers:	(a)	0	(e)	3/4	(i)	$2^{1/3}$
	(b)	1	(f)	3/5	(j)	$2^{-1/3}$
	(c)	2	(g)	21/80	(k)	$\sqrt{3}$
	(d)	3	(h)	3/20	(l)	None of the above

Questions: Use letters from answers above.

(i) A = ... ________

(ii) mean = ... ________

(iii) second moment ... ________

(iv) variance .... ________

(v) median .... ________

(vi) mode .... ________

(vii) range .... ________

Note: An answer can be used more than once. Only your answers above will be graded.

Problem 2:

A Bernoulli trial with success probability p is repeated until there is a failure. Let X be the number of trials. Find P[X=x].

Problem 3:

Θ is uniformly distributed on $(-\pi/2,\pi/2)$. $Y = \sin\Theta$. Find $f_Y(y)$.

Hint: $\frac{d}{dx}\arcsin x = (1-x^2)^{-\frac{1}{2}}$.

Problem 4:

A joint density, $f_{XY}(x,y)$, is equal to $x + y$ on the unit square ($0 \le x \le 1$, $0 \le y \le 1$) and is zero otherwise.

(a) Are X and Y independent? ☐ Yes

☐ No

(b) Compute $P[X < \frac{1}{2} \mid Y = 1]$

Problem 5:

X and Y are independent gamma random variables both with parameters b and c:

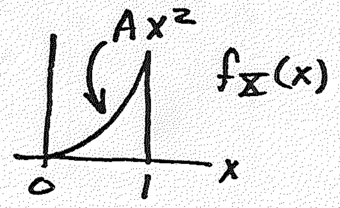
$$f_X(x) = f_Y(x) = \frac{c^{b+1}}{\Gamma(b+1)} x^b \exp(-cx)U(x)$$

Let $Z = X + Y$. Find $f_Z(z)$

Hint: $\Phi_X(\omega) = (1-jc\omega)^{-b-1}$

Solutions EE505 midterm 7/24/8

1\.



(i) $\int_0^1 Ax^2 dx = 1 = A\frac{1}{3}x^3\Big|_0^1 = \frac{A}{3} \Rightarrow A = 3$ $\Rightarrow$ _d_

(ii) $3\int_0^1 x^3 dx = 3\frac{1}{4} \Rightarrow$ _e_

(iii) $3\int_0^1 x^4 dx = \frac{3}{5}x^5\Big|_0^1 = 3/5 \Rightarrow$ _f_

(iv) $\sigma^2 = \frac{3}{5} - \frac{9}{16} = \frac{48-27}{80} = 21/80 \Rightarrow$ _g_

(v) $3\int_0^m x^2 dx = 3\int_m^1 x^2 dx = \frac{1}{2} \Rightarrow x^3\Big|_0^m = x^3\Big|_m^1$

$m^3 = 1 - m^3 \Rightarrow 2m^3 = 1 \Rightarrow m = (1/2)^{\frac{1}{3}} = 2^{-\frac{1}{3}} \Rightarrow$ _j_

(vi) Clearly, max is @ $x=1 \Rightarrow$ _b_

(vii) Clearly, 1 $\Rightarrow$ _b_

2\. $P_r[X=1] = q$ $P_r[X=4] = p^3 q$

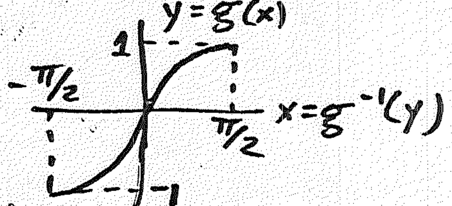
$P_r[X=2] = pq$ $\vdots$

$P_r[X=3] = p^2 q$ $P_r[X=x] = p^{x-1}q$ ⇐ geometric

3\. $Y = \sin\Theta \Rightarrow \Theta = g^{-1}(Y) = \sin^{-1} y$ ⇐ strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$

$$f_Y(y) = \frac{dg^{-1}(y)}{dy} f_\Theta(g^{-1}(y))$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & ; |\theta| < \pi/2 \\ 0 & ; \text{otherwise} \end{cases} \Rightarrow f_\Theta(\sin^{-1} y) = \begin{cases} \frac{1}{2\pi} & ; |\sin^{-1} y| < \pi/2 \\ 0 & ; \text{otherwise} \end{cases}$$



$-\frac{\pi}{2} < \sin^{-1} y < \frac{\pi}{2} \Rightarrow -1 < y < 1$

Thus $f_Y(y) = \begin{cases} \frac{1/\pi}{\sqrt{1-y^2}} & ; |y| < 1 \\ 0 & ; \text{otherwise} \end{cases}$

4(a) NO! (b) $f_X(x/y) = f_{XY}(x,y)/f_Y(y)$

$f_Y(y) = \int_0^1 (x+y)dx = \frac{x^2}{2} + yx\Big|_0^1 = \frac{1}{2} + y$

$\therefore f_X(x/Y) = \frac{x+y}{\frac{1}{2}+y}$ on the unit square given

$P[X < \frac{1}{2} / Y = 1] = \int_0^{1/2} \frac{x+y}{\frac{1}{2}+y} dx\Big|_{y=1} = \frac{2}{3}\int_0^{\frac{1}{2}}(x+1)dx$

$= \frac{2}{3}\left(\frac{x^2}{2} + x\right)\Big|_0^{1/2} = \frac{2}{3}\left(\frac{1}{8} + \frac{1}{2}\right) = \frac{2}{3}\cdot\frac{5}{8} = \frac{10}{24} = \frac{5}{12}$

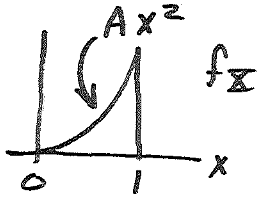
5\. $\Phi_Z(\omega) = \Phi_X(\omega)\Phi_Y(\omega) = (1-jc\omega)^{-2b-2} = (1-jc\omega)^{-(2b+1)-1}$

= characteristic function of gamma r.v. with parameters 2b+1 & c. Thus

$$f_Z(z) = \frac{c^{2b+2}}{\Gamma(2b+2)} x^{2b+1} e^{-cx} U(x)$$

Solutions EE505 midterm 7/24/84

1\.



(i) $\int_0^1 Ax^2 dx = 1 = A\frac{1}{3}x^3\Big|_0^1 = \frac{A}{3} \Rightarrow A = 3$ $\underline{\ d\ }$

(ii) $3\int_0^1 x^3 dx = 3\frac{1}{4} \Rightarrow \underline{\ e\ }$

(iii) $3\int_0^1 x^4 dx = \frac{3}{5}x^5\Big|_0^1 = 3/5 \Rightarrow \underline{\ f\ }$

(iv) $\sigma^2 = \frac{3}{5} - \frac{9}{16} = \frac{48-27}{80} = 21/80 \Rightarrow \underline{\ g\ }$

(v) $3\int_0^m x^2 dx = 3\int_m^1 x^2 dx = \frac{1}{2} \Rightarrow x^3\Big|_0^m = x^3\Big|_m^1$

$m^3 = 1 - m^3 \Rightarrow 2m^3 = 1 \Rightarrow m = (1/2)^{\frac{1}{3}} = 2^{-\frac{1}{3}} \Rightarrow \underline{\ j\ }$

(vi) Clearly, max is @ $x = 1 \Rightarrow \underline{\ b\ }$

(vii) Clearly, $1 \Rightarrow \underline{\ b\ }$

2\. $P_r[X = 1] = q$ $\qquad P_r[X = 4] = p^3 q$

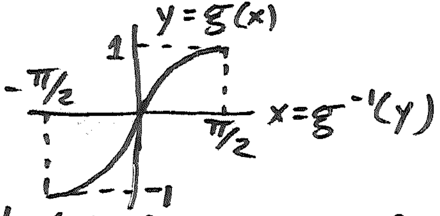
$P_r[X = 2] = pq$ $\qquad \vdots$

$P_r[X = 3] = p^2 q$ $\qquad P_r[X = x] = p^{x-1}q \Leftarrow$ geometric

3\. $Y = \sin\Theta \Rightarrow \Theta = g^{-1}(Y) = \sin^{-1} y \Leftarrow$ strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$

$$f_Y(y) = \frac{dg^{-1}(y)}{dy} f_\Theta(g^{-1}(y))$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{\pi} & ; |\theta| < \pi/2 \\ 0 & ; \text{otherwise} \end{cases} \Rightarrow f_\Theta(\sin^{-1}y) = \begin{cases} \frac{1}{2\pi} & ; |\sin^{-1}y| < \pi/2 \\ 0 & ; \text{otherwise} \end{cases}$$



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Thus $f_Y(y) = \begin{cases} \frac{1/\pi}{\sqrt{1-y^2}} & ; |y| < 1 \\ 0 & ; \text{otherwise} \end{cases}$

4(a) NO! (b) $f_X(x/y) = f_{XY}(x,y)/f_Y(y)$

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$\therefore f_X(x/Y) = \frac{x+y}{\frac{1}{2}+y}$ on the unit square given

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$= \frac{2}{3}\left(\frac{x^2}{2} + x\right)\Big|_0^{1/2} = \frac{2}{3}\left(\frac{1}{8} + \frac{1}{2}\right) = \frac{2}{3}\cdot\frac{5}{8} = \frac{10}{24} = \frac{5}{12}$

5\. $\Phi_Z(\omega) = \Phi_X(\omega)\Phi_Y(\omega) = (1 - jc\omega)^{-2b-2} = (1 - jc\omega)^{-(2b+1)-1}$

= characteristic function of gamma r.v. with parameters 2b+1 & c. Thus

$$f_Z(z) = \frac{c^{2b+2}}{\Gamma(2b+2)} x^{2b+1} e^{-cx} U(x)$$

2.7.

1. One fair coin is flipped 2 times. Are the 2 events

A: a head occurs on the first flip

B: a head occurs on the second flip

independent? YES

2. A fair coin is flipped 2 times. Let A be the event that a head occurs on the first flip and let B be the event that the same face does not occur on both flips. Are A and B independent? YES

3. An urn contains 4 balls numbered 1, 2, 3, 4, respectively. Two balls are drawn without replacement. Let A be the event that the first ball drawn has a 1 on it and let B be the event that the second ball has a 1 on it. Are A and B independent? NO

4. If the drawing is done with replacement in problem 3, are A and B independent? YES

5. A pair of dice is rolled 1 time. Let A be the event that the first die has a 1 on it, B the event that the second die has a 6 on it, and C the event that the sum is 7. Are A, B, and C independent? NO

6. A fair coin is flipped 3 times. Let A be the event that a head occurs on the first flip, let B be the event that at least 2 tails occur, and let C be the event that we get exactly 1 head or that we get tail, head, head in that order. Show that these 3 events satisfy equation 4 of Definition 2.7.3. but not equations 1, 2, or 3. [redacted]

7. Prove that if A and B are independent, so are $\bar{A}$ and $\bar{B}$.

8. The probability that a certain basketball player scores on a free throw is .7. If in a game he gets 15 free throws, compute the probability that he makes them all. Compute the probability that he makes 14 of them. What assumptions have you made in deriving your answer? $(0.7)^{15}$, $\binom{15}{4}(0.7)^{14}(0.3)$

9. Three teams, A, B, and C, enter a round-robin tournament. (Each team plays 2 games, 1 against each of the possible opponents. The winner of the tournament, if there is a winner, is the team winning both its games.) Assume that the game played is one in which a tie is not allowed. We assume the following probabilities:

$$P(A \text{ beats } B) = .7$$
$$P(B \text{ beats } C) = .8$$
$$P(C \text{ beats } A) = .9.$$

Compute the probability that team A wins the tournament; that team B wins the tournament. Compute the probability no one wins the tournament.

0.07, 0.24, 0.51

2.8.

1. A fair die is rolled until a 1 occurs. Compute the probability that:
   (a) 10 rolls are needed $\left(\frac{5}{6}\right)^9\left(\frac{1}{6}\right)$
   (b) less than 4 rolls are needed 91/216
   (c) an odd number of rolls is needed. 6/11

2. A fair pair of dice is rolled until a 7 occurs (as the sum of the 2 numbers on the dice). Compute the probability that
   (a) 2 rolls are needed 5/36
   (b) an even number of rolls is needed. 5/11

3. You fire a rifle at a target until you hit it. Assume the probability that you hit it is .9 for each shot and that the shots are independent. Compute the probability that:
   (a) it takes more than 2 shots. 0.01
   (b) the number of shots required is a multiple of 3. 1/111

4. Hugh takes a written driver's license test repeatedly until he passes it. Assume the probability that he passes it any given time is .1 and that the tests are independent. Compute the probability that:
   (a) it takes him more than 4 attempts $(0.9)^4$
   (b) it takes him more than 10 attempts. $(0.9)^{10}$

5. A traffic light on a route you travel every day turns red every 4 minutes, stays red 1 minute and then turns green again (thus it is green 3 minutes, red 1, etc.), with the red part of the signal starting on the hour, every hour.
   (a) If you arrive at the light at a random instant between 7:55 a.m. and 8:05 a.m., what is the probability that you have to stop at the light? 3/10
   (b) If you arrive at the light at a random instant between 7:54 a.m. and 8:04 a.m. what is the probability that you have to stop for the light? 2/10

6. The plug on an electric clock with a sweep second hand is pulled at a random instant of time within a certain minute. What is the probability that the second hand is between the 4 and the 5? Between the 1 and the 2? Between the 1 and the 6? $\frac{1}{12}, \frac{1}{12}, \frac{5}{12}$

7. A point is chosen at random between 0 and 1 on the x-axis in the (x, y) plane. A circle centered at the origin is then drawn in the plane, with radius determined by the chosen point. Compute the probability that the area of the circle is less than $\pi/2$. $\frac{1}{2}\sqrt{2}$

8. A 12-inch ruler is broken into 2 pieces at a random point along its length. What is the probability that the longer piece is at least twice the length of the shorter piece? 2/3

1

12. Given X has probability density function

$$f_X(x) = 1, \quad 99 < x < 100$$
$$= 0, \quad \text{otherwise},$$

derive $F_X(t)$.

$$F_X(t) = \begin{cases} 0 ; t<99 \\ t-99 ; 99<t<100 \\ 1 ; t>100 \end{cases}$$

13. Y is a continuous random variable with

$$f_Y(y) = 2(1-y), \quad 0 < y < 1$$
$$= 0, \quad \text{otherwise}.$$

Derive $F_Y(t)$.

$$\left[1-(1-t)^2\right] u(t)u(1-t)$$

14. Z is a continuous random variable with probability density function

$$f_Z(z) = 10e^{-10z}, \quad z > 0$$
$$= 0, \quad \text{otherwise}.$$

Derive $F_Z(t)$.

$$(1-e^{-10t})u(t)$$

EXERCISE 3.3.

1. If

$$p_X(x) = \tfrac{1}{4}, \quad x = 2, 4, 8, 16$$
$$= 0, \quad \text{otherwise}$$

compute:

(a) $E[X]$ 15/2
(b) $E[X^2]$ 85
(c) $E[1/X]$ 15/64
(d) $E[2^{X/2}]$ 139/2
(e) $\sigma_X{}^2$ and σ_X. , $\frac{115}{4}$

2. Suppose that $f_X(x) = \frac{1}{2}$, $-1 < x < 1$, compute:

(a) $E[X]$ 0
(b) $E[X^2]$ $\frac{1}{3}$
(c) $E[X+2]$ 2
(d) $E[X/4+7]$ 7
(e) $\sigma_X{}^2$ $\frac{1}{3}$
(f) σ_X.

3. Given

$$f_X(x) = 2(1-x), \quad 0 < x < 1$$
$$= 0, \quad \text{otherwise},$$

compute:

(a) $E[X]$ $\frac{1}{3}$
(b) $E[X^2]$ 1/6
(c) $E[(X+10)^2]$ $\frac{641}{6}$
(d) $E[1/(1-X)]$ 2
(e) $\sigma_X{}^2$ $\frac{1}{18}$
(f) σ_X.

4. Show that $E[X-\mu_X] = 0$.

6. Suppose that

$$F_U(t) = 0, \quad t < 1$$
$$= \log_e t, \quad 1 \le t \le e$$
$$= 1, \quad t > e,$$

find the median, $e^{1/2}$

7. If

$$F_Z(t) = 0, \quad t < 0$$
$$= 2^t - 1, \quad 0 \le t \le 1$$
$$= 1, \quad t > 1,$$

find the median. $\log_2 3/2$

10. The adult height of a 3-year-old boy is equally likely to fall in the interval from 5 feet 6 inches to 5 feet 11 inches. What is his expected height? 5 ' 8½ "

EXERCISE 3.5.

1. Let X be a random variable with distribution function $F_X(t)$ and let $Y = a + bX$ where $b < 0$. Derive the distribution function for Y.

2. Suppose that $b = 0$ in problem 1 above. Derive the distribution function for Y, defined as in that problem.

3. Given

$$F_X(t) = 0, \quad t < -1$$
$$= \frac{t+1}{2}, \quad -1 \le t \le 1$$
$$= 1, \quad t > 1,$$

$f_Y(t) = \frac{1}{4}$; $13 \le t \le 17$

find the distribution function for $Y = 15 + 2X$ and the density function for Y.

4. Suppose that

$$F_W(t) = 0, \quad t < 0$$
$$= t^3, \quad 0 \le t \le 1$$
$$= 1, \quad t > 1$$

and let $Z = W - 1$. Find $F_Z(t)$ and $f_Z(t)$. $f_Z(t) = 3(t+1)^2; \; -1 \le t \le 0$

2

5. If

$$F_X(t) = \begin{cases} 0, & t < -10 \\ \frac{1}{4}, & -10 \le t < 0 \\ \frac{3}{4}, & 0 \le t < 10 \\ 1, & t \ge 10, \end{cases}$$

$$f_{\text{U}}(t) = \frac{1}{4}\,\delta(t+120) + \frac{1}{2}\,\delta(t+50) + \frac{1}{4}\,\delta(t-20)$$

find the distribution function for

$$U = 7X - 50 \text{ and } p_U(u).$$

6. If

$$F_Y(t) = \begin{cases} 1 - e^{-t}, & t \ge 0 \\ 0, & t < 0, \end{cases}$$

find $F_X(t)$ and $f_X(t)$ where $X = 2Y - 7$.

10. Let

$$F_X(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t \le 1 \\ 1, & t > 1, \end{cases}$$

$\frac{1}{2\sqrt{t}}\ ; 0 \le t \le 1$

and find $F_Z(t)$ and $f_Z(t)$ where $Z = X^2$

11. If

$$F_X(t) = \begin{cases} 0, & t < -1 \\ \frac{t+1}{2}, & -1 \le t \le 1 \\ 1, & t > 1, \end{cases}$$

EXERCISE 4.3.

1. It has been observed that cars pass a certain point on a rural road at the average rate of 3 per hour. Assume that the instants at which the cars pass are independent and let X be the number that pass this point in a 30-minute interval. Compute $P(X = 0)$, $P(X \ge 2)$. 0.2231 0.4422

2. It has been observed empirically that deaths per hour, due to traffic accidents, occur at a rate of 8 per hour on long holiday weekends in the United States. Assuming that these deaths occur independently, compute the probability that a 1-hour period would pass with no deaths; that a 15-minute period would pass with no deaths; that 4 consecutive, nonoverlapping 15-minute periods would pass with no deaths. 0.0003, 0.1353, 0.0003

3. It has been observed that packages of Hamm's beer are removed from the shelf of a particular supermarket at a rate of 10 per hour during rush periods. What is the probability that at least 1 package is removed during the first 10 minutes of a rush period? What is the probability that at least 1 is removed from the shelf during each of 3 consecutive, nonoverlapping 10-minute intervals? 0.811, $(\ )^3$

4. At a certain manufacturing plant, accidents have been occurring at the rate of 1 every 2 months. Assuming that the accidents occur independently, what is the expected number of accidents per year? What is the standard deviation of the number of accidents per year? What is the probability of there being no accidents in a given month? 6, $\sqrt{6}$, 0.6065

EXERCISE 4.4.

1. Suppose that X is uniformly distributed on the interval $(1, 2)$ and we construct a square having sides of length X. Derive the probability density function of $Y = X^2$, the area of the square, and compute $P(Y > 2)$. $1/2\sqrt{t}$, $2 - \sqrt{2}$

2. If X is uniformly distributed on the interval $(1, 4)$, derive the density function of $Z = X^{1/2}$. $2t/3$

5. Suppose that quarter-pound bars of butter are cut from larger slabs by a machine. We assume that the larger slabs are quite uniform in density; if the length of the bar is exactly $3\frac{3}{8}$ inches, then the bar will weigh $\frac{1}{4}$ pound. Suppose that the true length X of a bar cut by this machine is equally likely to lie in the interval from 3.35 inches to 3.45 inches. Assuming that the lengths of bars cut by this machine are independent, what is the probability that all 4 bars in a particular pound package of butter will weigh at least $\frac{1}{4}$ pound? That exactly 3 will weigh at least $\frac{1}{4}$ pound? 0.316, 0.422

$F_{\text{Y}}(t) = (1 - e^{-\lambda t})$

6. X is uniformly distributed on $(0, 2)$ and Y is exponential with parameter λ. Find the value of λ such that $P(X < 1) = P(Y < 1)$. 0.69

7. Calls arrive at a switchboard according to a Poisson process with parameter $\lambda = 5$ per hour. If we are at the switchboard, what is the probability that it is at least 15 minutes until the next call? That it is no more than 10 minutes? That it is exactly 5 minutes?

8. A newsboy is selling papers on a busy street. The papers he sells are events in a Poisson process with parameter $\lambda = 50$ per hour. If we have just purchased a paper from him, what is the probability that it will be at least 2 minutes until he sells another? If it is already 5 minutes since his last sale, what is the probability it will be at least 2 more minutes until his next sale? 0.1882, 0.1882

9. X is uniform on $(-1, 3)$ and Y is exponential with parameter λ. Find λ such that $\sigma_X^2 = \sigma_Y^2$. $\frac{1}{2}\sqrt{3}$

10. X is geometric with parameter p and Y is exponential with parameter λ. Find λ such that $P(X > 1) = P(Y > 1)$. $-\ln(1-p)$

11. We are given a Poisson process with parameter λ. We begin observing the process at time zero; let S be the time until the second event occurs. Derive the probability density function for S. $(1-2t)^{-1/2}$

3

EX ISE 4.5.

1. Assume that the time X required for a distance runner to run a mile is a normal random variable with parameters $\mu = 4$ minutes, 1 second and $\sigma = 2$ seconds. What is the probability that this athlete will run the mile in less than 4 minutes? In more than 3 minutes, 55 seconds?

0.3085, 0.9987

2. The length X of an adult rock cod caught in Monterey Bay is a normal random variable with parameters $\mu = 16$ inches and $\sigma = 1$ inch. If you catch one of these fish, what is the probability that it will be at least 14 inches long? That it will be no more than 17 inches long? That its length will be between 12 inches and 15 inches? 0.9773, 0.8413, 0.1587

3. If Z is a standard normal random variable and we define $U = |Z|$, then U is called the folded standard normal variable. Express $F_U(t)$ in terms of $F_Z(t)$.

$[2F_Z(t) - 1]u(t)$

4. Suppose that we are given a target with a vertical straight line drawn through its center. Let us assume that if we throw a dart at this target and measure the distance Z between the point we hit and the center line, then Z is a standard normal random variable (if the dart lands right of the center line the measurement is positive, if it lands to the left of the center line the measurement is negative). Then, the distance from the point we hit to the center line is $|Z| = U$, the folded normal random variable defined in problem 3. Compute $P(U > 1)$ and $P(U < \frac{1}{2})$. 0.3147, 0.3830

7. Show that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx = 1$$

for any μ and for $\sigma > 0$. (Hint: If

$$A = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx,$$

then

$$A^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-(x-\mu)^2/2\sigma^2-(y-\mu)^2/2\sigma^2}\, dx\, dy;$$

let $u = (x - \mu)/\sigma$, $v = (y - \mu)/\sigma$, and transform to polar coordinates to show $A^2 = 1$ which implies $A = 1$.)

EXERCISE 5.5.

1. If X is uniformly distributed on the interval (0, 1), compare $P(|X - \mu_X| < k\sigma_X)$ with the values given by the Chebychev inequality for $k = 1\frac{1}{4}, 1\frac{1}{2}, 1\frac{3}{4}$, and 2.

2. For any value of $k \geq 1$, we can define a discrete random variable X to have probability function

$$\begin{aligned} p_X(x) &= \frac{k^2-1}{k^2}, & x &= 0 \\ &= \frac{1}{2k^2}, & x &= -k, k \\ &= 0, & &\text{otherwise.} \end{aligned}$$

Compute μ_X and σ_X, and compare the exact probability $P(|X - \mu_X| < k\sigma_X)$ with the bound given by Chebychev's inequality.

4

Solutions

1. Three dice are rolled. Consider the following events:

A: The outcome on the blue die is odd
B: The outcome on the red die is even
C: The outcome on the yellow die is one
D: The sum of the red and yellow dice is even
E: The sum of the red and yellow dice is four
F: The sum of the red and yellow dice is five
G: The sum of the blue and yellow dice is three
H: The sum of the blue and yellow dice is two
I: The sum of all three dice is three
J: You will pass the test

Using the notation:
m = mutually exclusive
i = independent
s = one event is a subset of the other
n = none of the above

classify the following event pairs:

A and B ______ i
A and C ______ i
B and D ______ i since $Pr[D/B] = Pr[D] = \frac{1}{2}$
C and D ______ i
D and E ______ s
C and H ______ s
C and G ______ n
C and I ______ s
H and I ______ s
A and J ______ i

Compute the distribution function, $F_X(x)$, for a Rayliegh random variable with parameter α.

$$f_X(x) = \frac{x}{\alpha^2} e^{\frac{-x^2}{2\alpha^2}} U(x)$$

$$F_X(x) = \frac{1}{\alpha^2} \int_0^x \xi e^{-\xi^2/2\alpha^2} d\xi$$

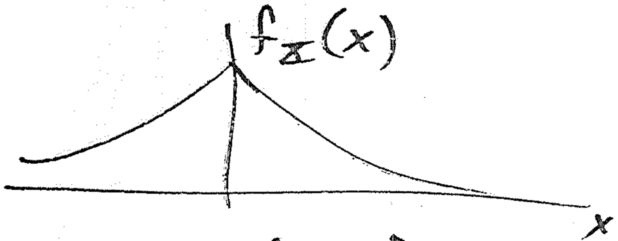
$$= \frac{-c}{\alpha^2} e^{-\xi^2/2\alpha^2} \Big|_0^x \quad ; \quad c = \text{constant}$$

$$= \frac{c}{\alpha^2} \left[1 - e^{-x^2/2\alpha^2}\right] U(x)$$

Since $F_X(\infty) = 1$

$$F_X(x) = \left[1 - e^{-x^2/2\alpha^2}\right] U(x)$$

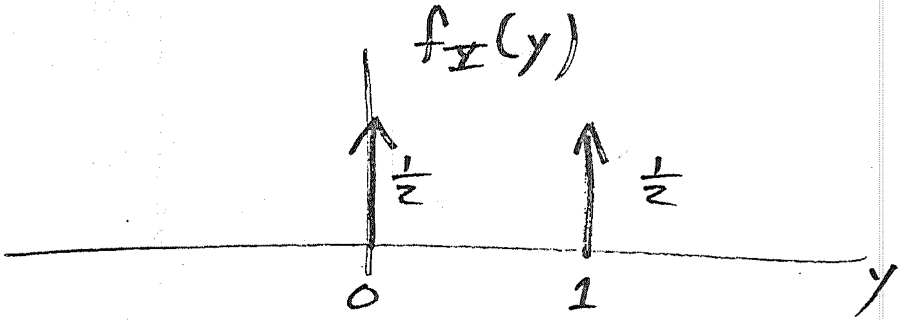
Let X be a Laplace random variable. Define Y = U(X) where U() is the unit step. Find the density function for Y.



Clearly, if $X \geq 0$, $Y = 1$

if $X < 0$, $Y = 0$

Thus, half the mass goes to zero and half to one:



or

$$f_Y(y) = \frac{1}{2}\delta(y) + \frac{1}{2}\delta(y-1)$$

CONSIDER THE TRUNCATED CAUCHY DENSITY:

$$f_X(x) = \begin{cases} A(x^2+1)^{-1} & ; \ |x| \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

(a) Compute A
(b) Compute E(X)
(c) Compute var(X)
(d) Compute the conditional density, $f_X(x/ X>0)$.

a. $\int_{-1}^{1} f_X(x)dx = 1 = A\int_{-1}^{1} \frac{dx}{x^2+1}$

$= A \tan^{-1}x \Big|_{-1}^{1} = A\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi A}{2} \Rightarrow A = \frac{2}{\pi}$

b. $E[X] = 0$ since f_X is even

c. $\operatorname{var} X = E[X^2] = \frac{2}{\pi}\int_{-1}^{1}\frac{x^2 dx}{x^2+1}$

$= \frac{2}{\pi}\left[\xi - \tan^{-1}\xi\right]_{-1}^{1}$

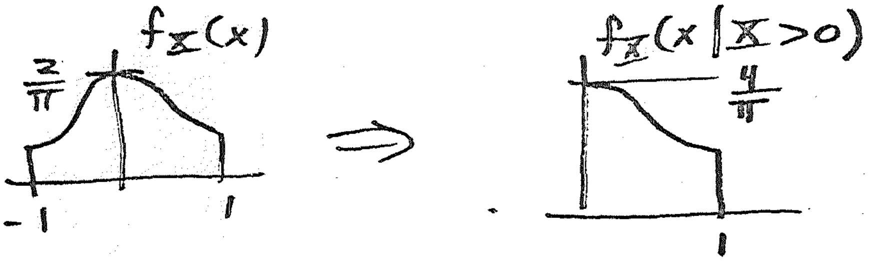
$= \frac{2}{\pi}\left[(1-\tan^{-1}1) - (-1-\tan^{-1}-1)\right]$

$= \frac{2}{\pi}\left[\left(1-\frac{\pi}{4}\right) - \left(-1+\frac{\pi}{4}\right)\right]$

$= \frac{2}{\pi}\left[1 - \frac{\pi}{4} + 1 - \frac{\pi}{4}\right] = \frac{2}{\pi}\left[2 - \frac{\pi}{2}\right] = \frac{4}{\pi} - 1$

$= 0.27324$

d.



$$f_X(x|X>0) = \begin{cases} \frac{4/\pi}{x^2+1} & ; \ 0 \le x \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

Four major prizes are to be given away in a state lottery. Four drawings are made without replacement from all entries. Thus, one entry can at most win one prize. You enter the lottery six times. The total number of entries (including yourself) is 10,000. What is the probability you will win at least one of the major prizes? Give a number for your answer.

$$P[\text{win}] = 1 - P[\text{loose}]$$

$$P[\text{loose}] = \frac{9994}{\underset{\substack{\nearrow \\ \text{First} \\ \text{draw}}}{10{,}000}} \cdot \frac{9993}{\underset{\substack{\nearrow \\ \text{Second}}}{9999}} \cdot \frac{9992}{\underset{\substack{\nearrow \\ \text{Third}}}{9998}} \cdot \frac{9991}{\underset{\substack{\nearrow \\ \text{Fourth}}}{9997}}$$

$$= 0.99760$$

$$\Rightarrow Pr[\text{win}] = 0.00240$$

You perform a Bernoulli trial. The chance of success is p. You perform the trial until you get a failure. Let N be the random variable equal to the number of trials performed.

(a) What is Pr(N=m) for some given m?
(b) Find the pdf, $f_N(x)$.
(C) Compute E(N).

(a) $Pr[N=1] = q$
$Pr[N=2] = pq$
$Pr[N=3] = p^2 q$
$\vdots$
$Pr[N=m] = p^{m-1} q$

(b) $f_N(x) = \sum_{k=1}^{\infty} p^{k-1} q \, \delta(x-k)$

(c) $E[X] = q \sum_{k=1}^{\infty} k p^{k-1}$

$$\frac{E[X]}{q} = \sum_{k=1}^{\infty} k p^{k-1} = \sum_{k=0}^{\infty} k p^{k-1}$$

$$\int^{p} \frac{E[X]}{q} dp = \sum_{k=0}^{\infty} p^k + \text{const}$$

$$= \frac{1}{1-p} + \text{const}$$

$$\frac{E[X]}{q} = \frac{1}{(1-p)^2} = \sum_{k=1}^{\infty} k p^{k-1} \Leftarrow \text{Given as "Hint"}$$

$$E[X] = \frac{q}{(1-p)^2} = \frac{1}{q}$$

Note: For coin, = 2

A random variable has unit variance and zero mean. You wish to set a threshold, T, so that the probability of the magnitude of the random variable exceeding T is no greater than one chance in a hundred. What is a good value for T? Justify your choice.

Tchebycheff's $\neq$ for zero mean, $\sigma = 1$;

$$Pr[|X| \geq k] \leq \frac{1}{k^2} = \frac{1}{100}$$

$k = 10$

Thus set $T = 10$

QUIZ ONE

NAME ____________________________

EE505
Summer 83
Bob Marks

SCORE ____________________ / 175

1. Three dice are rolled. Consider the following events:

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B: The outcome on the red die is even
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H: The sum of the blue and yellow dice is two
I: The sum of all three dice is three
J: You will pass the test

Using the notation:
m = mutually exclusive
i = independent
s = one event is a subset of the other
n = none of the above

classify the following event pairs:

A and B ______________

A and C ______________

B and D ______________

C and D ______________

D and E ______________

C and H ______________

C and G ______________

C and I ______________

H and I ______________

A and J ______________

Helpful hints to use elsewhere on this test:

$$\arctan \pm 1 = \pm \frac{\pi}{4}$$

$$\frac{d}{dx} e^{x^2} = 2x e^{x^2}$$

$$\int \frac{dx}{x^2+1} = \arctan x$$

$$\sum_{k=1}^{\infty} k a^{k-1} = (a-1)^{-2} ; |a|<1$$

$$\sum_{k=0}^{\infty} a^k = (1-a)^{-1} ; |a|<1$$

Compute the distribution function, $F_X(x)$, for a Rayliegh random variable with parameter α.

Let X be a Laplace random variable. Define Y = U(X) where U() is the unit step. Find the density function for Y.

CONSIDER THE TRUNCATED CAUCHY DENSITY:

$$f_X(x) = \begin{cases} A(x^2+1)^{-1} & ; |x| \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$

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You perform a Bernoulli trial. The chance of success is p. You perform the trial until you get a failure. Let N be the random variable equal to the number of trials performed.

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A random variable has unit variance and zero mean. You wish to set a threshold, T, so that the probability of the magnitude of the random variable exceeding T is no greater than one chance in a hundred. What is a good value for T? Justify your choice.

[illegible] Final Examination Summer '83

Solutions

1. A random variable X has a characteristic function:

$$\Phi_X(\omega) = A\cos^2(a\omega)$$

where a is a given parameter. Compute:

(a) A
(b) E(X)
(c) var(X)

(a) $\Phi_X(0) = 1 \Longrightarrow A = 1$

(b) $\Phi_X(\omega) = \frac{1}{2}(1 + \cos 2a\omega)$

$$\frac{d\Phi_X}{d\omega} = -\frac{1}{2}\,2a\sin 2a\omega = -a\sin 2a\omega$$

$$\frac{d\Phi_X(0)}{d\omega} = jE[X] = 0 \Longrightarrow E(X) = 0$$

(c) $\frac{d^2\Phi_X}{d\omega^2} = -a(2a)\cos 2a\omega$

$$\frac{d^2\Phi_X(0)}{d\omega^2} = j^2E(X^2) = -2a^2 \Longrightarrow E(X^2) = 2a^2$$

$$\operatorname{var} X = E(X^2) - E^2(X)$$
$$= 2a^2$$

-2-

Sometimes the expected value of a random variable is not such a good estimate. For example, let X be a Poisson random variable with parameter a=1. Let

$$Y = (-1)^X$$

Compute E(Y) and comment.

$$E[Y] = \sum_{k=0}^{\infty} (-1)^k \frac{e^{-a} a^k}{k!} \quad ; a = 1$$

$$= e^{-1} \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} = e^{-1} e^{-a}$$

$$= e^{-2}$$

$$= 0.135$$

COMMENT: Y is always ±1

-3-

Let X be a gamma random variable with n an integer:

$$f_X(x) = \frac{c^{n+1}}{n!} x^n \exp(-cx)\, U(x)$$

Let $\{X_n | n=1,2,\ldots,N\}$ be iid samples from this density. Define

$$\overline{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$$

Find the density function for the average, $f_{\overline{X}}(x)$.

$$\Phi_{\overline{X}}(\omega) = E\left[e^{j\omega\overline{X}}\right] = E\left[e^{j\frac{\omega}{N}\sum_{n=1}^{N} X_n}\right]$$

$$= E\left[\prod_{n=1}^{N} e^{j\omega X_n/N}\right]$$

$$= \prod_{n=1}^{N} E\left[e^{j\omega X_n/N}\right]$$

$$= E^N\left[e^{j\omega X_n/N}\right]$$

$$= \Phi_X^N\left(\frac{\omega}{N}\right)$$

From p.159 of text:

$$\Phi_X(\omega) = \frac{c^{n+1}}{(c-j\omega)^{n+1}}$$

$$\Rightarrow \Phi_{\overline{X}}(\omega) = \frac{c^{N(n+1)}}{\left(c - j\frac{\omega}{N}\right)^{N(n+1)}} \quad ; \quad \hat{n}+1 = N(n+1) = (\underbrace{Nn+N-1}_{\hat{n}})+1$$

From Fourier xform scaling theorem

$$f_{\overline{X}}(x) = N \frac{c^{\hat{n}+1}}{\hat{n}!} (Nx)^{\hat{n}} e^{-c(Nx)}\, \mathcal{U}(x)$$

$$= \frac{N\, c^{N(n+1)}}{(Nn+N-1)!} (Nx)^{Nn+N-1} e^{-cNx}\, \mathcal{U}(x)$$

= [illegible]

Let P and Q denote independent random variables both uniformly distribut d on the interval from zero to unity. Define the process:

$$X(t) = P\,e^{-Qt}\,U(t)$$

Find:

(a) $E(X(t))$

(b) $R(t_1,t_2)$

(c) $\text{var } X(t)$

(a) $E[X(t)] = E[Pe^{-Qt}U(t)]$

$= E[P]E[e^{-Qt}]u(t)$; $E[P] = \frac{1}{2}$

$E[e^{-Qt}] = \int_0^1 e^{-qt}\,dq = -\frac{1}{t}e^{-qt}\Big|_0^1 = \frac{1-e^{-t}}{t}$

$\Rightarrow E[X(t)] = \frac{1-e^{-t}}{2t}\,U(t)$

(b) $R(t_1,t_2) = E[X(t_1)X(t_2)] = E[P^2 e^{-Q(t_1+t_2)}]U(t_1)U(t_2)$

$= E[P^2]\,E[e^{-Q(t_1+t_2)}]\,U(t_1)\,U(t_2)$

$E[P^2] = \int_0^1 p^2\,dp = \frac{1}{3}$

$E[e^{-Q(t_1+t_2)}] = \frac{1-e^{-(t_1+t_2)}}{t_1+t_2}$

$\Rightarrow R(t_1,t_2) = \frac{1-e^{-(t_1+t_2)}}{3(t_1+t_2)}\,U(t_1)U(t_2)$

(c) $E[X^2(t)] = R(t,t) = \frac{1-e^{-2t}}{6t}\,U(t)$

$\text{var } X(t) = E[X^2] - E(X)^2$

$= \left(\frac{1-e^{-2t}}{6t} - \left(\frac{1-e^{-t}}{2t}\right)^2\right)U(t)$

We draw N iid samples from a shifted Laplacian random variable with mean η and variance σ^2. Give an approximation of the density function for the average of these numbers. $N >> 1$

Central limit theorem:

$$\overline{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$$

$$E[\overline{X}] = E[X] = \eta$$

$$\text{var}\,\overline{X} = \frac{\sigma^2}{N}$$

$\Rightarrow \overline{X}$ in normal (mean $= \eta$, variance $= \frac{\sigma^2}{N}$)

or

$$\overline{X} \sim \frac{\sqrt{N}}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-\eta)^2}{2\sigma^2/N}}$$

X(t) is a stationary random process with mean η and autocorrelation

$$R_x(\tau) = \overline{x^2}\ \exp\ -a|\tau|$$

where a is a specified parameter. What percentage of the time can w
expect X(t) to lie below a given threshold, T?

$$P_r[X(t) \le T] = ?\ ,\quad var = \sqrt{\overline{x^2} - \eta^2}$$

$$X(t) \sim \frac{1}{\sqrt{2\pi}\sqrt{var}}\, e^{-\frac{(x-\eta)^2}{2\,var}}$$

$$P_r[X(t) \le T] = \int_{-\infty}^{T} \frac{1}{\sqrt{2\pi}\,var}\, e^{-\frac{(x-\eta)^2}{2\,var}}\, dx$$

$$\text{Let}\quad Y = \frac{x-\eta}{\sqrt{var}}$$

$$P_r[X(t) \le T] = \int_{-\infty}^{\frac{T-\eta}{\sqrt{var}}} \frac{1}{\sqrt{2\pi}}\, e^{-Y^2/2}\, dY$$

$$= \frac{1}{2} + erf\ \frac{T-\eta}{\sqrt{var}}$$

$$= \frac{1}{2} + erf\left(\frac{T-\eta}{\sqrt{\overline{x^2} - \eta^2}}\right)$$

In our take-home problem last week, we found that the joint density

$$f_{XY}(x,y) = 8\, y^2\, e^{-2y}\, e^{-2xy}\, U(x)\, U(y)$$

had a marginal density

$$f_Y(y) = 4\, y\, e^{-2y}\, U(y).$$

Given that $Y=2$, what is a good estimate of X?

minimum mse is $E[X/Y]$

$$f_X(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$= \frac{8y^{\cancel{2}}\cancel{e^{-2y}}e^{-2xy}}{4\cancel{y}\cancel{e^{-2y}}}$$

$$= 2y\, e^{-2xy}\, U(x)\, U(y)$$

$$E[X/Y] = 2y\int_0^\infty x e^{-2xy}\,dx\; U(y)$$

$$u = x \quad dv = e^{-2xy}\,dx$$

$$du = dx \quad v = \frac{-1}{2y} e^{-2xy}$$

$$= 2y\left[\underset{0}{\cancel{\frac{-x}{2y} e^{-2xy}\Big|_0^\infty}} + \frac{1}{2y}\int_0^\infty e^{-2xy}\,dx\right] U(y)$$

$$= 2y\left[\frac{1}{2y}\cdot\frac{1}{2y}\right] = \frac{1}{2y}$$

Choose $\hat{X} = \frac{1}{4}$

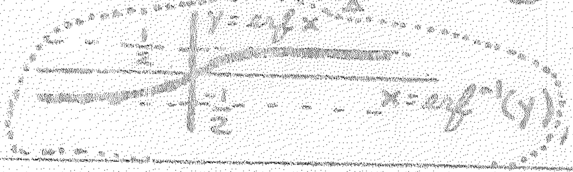
NAME____________________

useful relationships:

Beta density: $f_X(x) = \frac{1}{B(b+1,c+1)} x^b (1-x)^c$ on $0 \le x \le 1$

Normal Density: $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp[(x-m)^2/2\sigma^2]$ on $-\infty < x < \infty$

Poisson density: $f_X(x) = e^{-a} \sum_{k=0}^{\infty} [a^k/k!]\,\delta(x-k)$ | Gaussian: $F_X(x) = \frac{1}{2} + \mathrm{erf}\left(\frac{x-}{\sigma}\right.$



$$\mathrm{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-u^2/2}\,du$$

$B(b,c) = \Gamma(b)\Gamma(c)/\Gamma(b+c)$; $\Gamma(n+1) = n!$

1. A random variable has a probability density function:

$$f_X(x) = \begin{cases} A/x^2 & ; 1 \le x \le 2 \\ 0 & ; \text{otherwise} \end{cases}$$

Compute:
(a) A
(b) E(X)
(c) var(X)

2. A system consists of two lightbulbs connected in series. If the system fails we assume one of the two bulbs failed. The probability that bulb A will work at time t is $P_A(t) = \exp(-t)\, U(t)$. Similarly, for bulb B: $P_B(t) = \exp(-2t)\, U(t)$. If the system fails in 4 time-units, what is the probability that bulb A caused the failure?

3. The random variable X is distributed Poisson with parameter a. Find the conditional density function if we know that $0 \leq X \leq 2$

4. X is uniformly distributed between zero and unity. We perform the transformation:

$$Y = \ln(X)$$

Find the density function for the random variable, Y.

5. X is distributed as a Beta random variable.

(a) Compute $E(X^n)$

(b) Simplify your solution to a ratio of products of factorials when b & c are integers.

6. It is easy to generate a uniformly distributed random variable on a computer. Many engineering problems require normal (or Gaussian) random variables. Compute a non-linearity, $g(x)$, such that

$$Y = g(X)$$

is a normal random variable with mean η and variance σ^2 when X is uniformly distributed between zero and unity.

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SOLUTIONS!

Some useful relationships:

Beta density: $f_X(x) = \frac{1}{B(b+1, c+1)} x^b (1-x)^c$ on $0 \le x \le 1$

Normal Density: $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp[(x-\eta)^2/2\sigma^2]$ on $-\infty < x < \infty$

Poisson density: $f_X(x) = e^{-a} \sum_{k=0}^{\infty} [a^k/k!]\, \delta(x-k)$ | Gaussian: $F_X(x) = \frac{1}{2} + \mathrm{erf}\left(\frac{x-\eta}{\sigma}\right.$

$$\mathrm{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-u^2/2}\, du$$

$$B(b,c) = \Gamma(b)\Gamma(c)/\Gamma(b+c) \;;\; \Gamma(n+1) = n!$$

1. A random variable has a probability density function:

$$f_X(x) = \begin{cases} A/x^2 & ; \; 1 \le x \le 2 \\ 0 & ; \; \text{otherwise} \end{cases}$$

Compute:
(a) A
(b) E(X)
(c) var(X)

(a) $\int_1^2 f_X(x)\,dx = 1 = A \int_1^2 x^{-2} = A \left(-x^{-1}\right)\Big|_1^2$

$= A\left(-\frac{1}{2}+1\right) = A\,\frac{1}{2} \Rightarrow A = 2$

(b) $E(x) = 2\int_1^2 x \left(\frac{1}{x^2}\right) dx = 2\int_1^2 \frac{dx}{x} = 2 \ln x \Big|_1^2 = 2 \ln 2$

(c) $E(x^2) = 2\int_1^2 x^2 \frac{1}{x^2}\, dx = 2$

$\mathrm{var}\, x = E(x^2) - E^2 x$

$= 4 - 4 \ln^2 2$

$= 4(1 - \ln^2 2) \simeq 2.078$

2. A system consists of two lightbulbs connected in series. If the system fails, we assume one of the two bulbs failed. The probability that bulb A will work at time t is $P_A(t) = \exp(-t)\, U(t)$. Similarly, for bulb B: $P_B(t) = \exp(-2t)\, U(t)$. If the system fails in 4 time-units, what is the probability that bulb A caused the failure?

CAN USE BAYES Theorem:

$$P(A/F) = \frac{P(F/A)\,P(A)}{P(F/A)\,P(A) + P[F/B]\,P(B)}$$

A = A FAILED, B = B FAILED

F = SYSTEM FAILED

OBVIOUSLY: $P[F/A] = P[F/B] = 1$

$$P(A) = 1 - P_A(4) = 1 - e^{-4}$$

$$P(B) = 1 - P_B(4) = 1 - e^{-8}$$

$$P(A/F) = \frac{1 - e^{-4}}{(1-e^{-4}) + (1-e^{-8})}$$

$$\cong 0.495$$

3. The random variable X is distributed Poisson with parameter a. Find the conditional density function if we know that $0 \le X \le 2$

$$X \sim f_X(x) = \sum_{k=0}^{\infty} \frac{e^{-a} a^k}{k!} \delta(x-k)$$

NOTE:

$$f_X(x \mid 0 \le X \le 2) \equiv f_X(x \mid X \le 2)$$

$$= \begin{cases} \dfrac{f_X(x)}{F_X(2)} & ; x \le 2 \\ 0 & ; x > 2 \end{cases} \Leftarrow \text{Derived in class}$$

$$F_X(2) = P_r[X \le 2]$$

$$= \left(1 + a + \frac{a^2}{2}\right) e^{-a}$$

$$f_X(x \mid 0 \le X \le 2) = \frac{e^{a}}{1 + a + \frac{a^2}{2}} \times \Big[e^{-a} \delta(x)$$

$$+ e^{-a} a \, \delta(x-1) + e^{-a} \frac{a^2}{2} \delta(x-2) \Big]$$

$$= \frac{\delta(x) + a\,\delta(x-1) + \frac{a^2}{2} \delta(x-2)}{1 + a + \frac{a^2}{2}}$$

4. X is uniformly distributed between zero and unity. We perform the transformation:

$$Y = \ln(X)$$

Find the density function for the random variable, Y.

$y = g(x) = \ln x \Rightarrow x = g^{-1}(y) = e^{y}$

Note, $\ln x$ is monotonically $\uparrow$ over $0 \le x \le 1$.

$$\Rightarrow f_Y(y) = \frac{dg^{-1}(y)}{dy} f_X(g^{-1}(y))$$

$$= e^{y} f_X(e^{y})$$

$f_X(e^{y}) = ?$

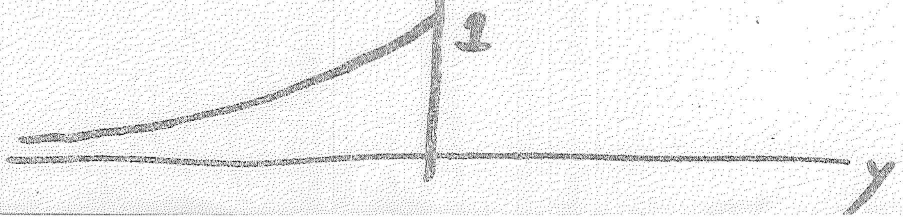
$$f_X(x) = \begin{cases} 1 & ; 0 \le x \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$

$$\therefore f_X(e^{y}) = \begin{cases} 1 & ; 0 \le e^{y} \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & ; -\infty \le y \le 0 \\ 0 & ; \text{otherwise} \end{cases}$$

Thus:

$$f_Y(y) = \begin{cases} e^{y} & ; -\infty \le y \le 0 \\ 0 & ; y > 0 \end{cases}$$



5. X is distributed as a Beta random variable.

(a) Compute $E(X^n)$

(b) Simplify your solution to a ratio of products of factorials when b ; c are integers.

$$f_X(x) = B(b+1,c+1)\, x^b (1-x)^c \quad ; \quad 0 \le x \le 1$$

Note: $\int_0^1 f_X(x) = 1$

Thus: $\frac{1}{B(b+1,c+1)} = \int_0^1 x^b (1-x)^c\, dx$

Now:

$$E[x^n] = B(b+1,c+1)\int_0^1 x^{b+n}(1-x)^c\, dx$$

$$= B(b+1,c+1)\,\frac{1}{B(b+n+1,c+1)}$$

$$= \frac{B(b+1,c+1)}{B(b+n+1,c+1)}$$

$$= \frac{\Gamma(b+1)\cancel{\Gamma(c+1)}}{\Gamma(b+c+2)} \times \frac{\Gamma(b+c+n+2)}{\Gamma(b+n+1)\cancel{\Gamma(c+1)}}$$

$$= \frac{b!\,(b+c+n+1)!}{(b+c+1)!\,(b+n)!}$$

if b ; c are integers

6. It is easy to generate a uniformly distributed random variable on a computer. Many engineering problems require normal (or Gaussian) random variables. Compute a non-linearity, g(x), such that

$$Y = g(X)$$

is a normal random variable with mean η and variance σ^2 when X is uniformly distributed between zero and unity.

Assume g is strictly increasing. If so:

$$F_Y(y) = F_X\left[g^{-1}(y)\right]$$

where:

$$F_X(x) = \begin{cases} 0 & ; x < 0 \\ x & ; 0 \le x \le 1 \\ 1 & ; x \ge 1 \end{cases}$$

thus:

$$F_X\left[g^{-1}(y)\right] = \frac{1}{2} + \operatorname{erf}\left(\frac{y-\eta}{\sigma}\right)$$

$$= \begin{cases} 0 & ; g^{-1}(y) \le 0 \\ g^{-1}(y) & ; 0 \le g^{-1}(y) \le 1 \\ 1 & ; g^{-1}(y) \ge 1 \end{cases}$$

<u>Must</u> be

$$x = g^{-1}(y) = \frac{1}{2} + \operatorname{erf}\left(\frac{y-\eta}{\sigma}\right) \longrightarrow$$

ALWAY'S between 0 & 1

Solving for $y = g(x)$:

$$\operatorname{erf}\left(\frac{y-\eta}{\sigma}\right) = x - \frac{1}{2}$$

$$y = g(x) = \sigma \operatorname{erf}^{-1}\left(x - \frac{1}{2}\right) + \eta$$

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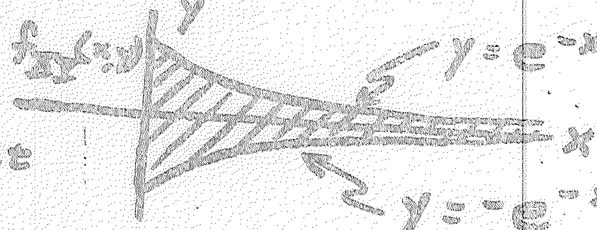
Course Grade = /4.0

1. A joint pdf has uniform mass over the shown shaded area:



(a) What is the uniform height of $f_{XY}(x,y)$?

(b) Compute the marginal pdf, $f_Y(y)$.

2. A sample Y is taken from a normal pdf with mean m_y and variance σ^2. Form the random process:

$$X(t) = Y \quad ; -\infty < t < \infty$$

(a) Compute the mean and autocorrelation of $X(t)$. Is the process stationary?

(b) $X(t)$ is the input to a linear time-invariant filter with impulse response $h(t) = e^{-\alpha|t|}$. Let $Z(t)$ denote the filter output. Compute the mean & autocorrelation of $Z(t)$.

3. LET $X(t)$ DENOTE A ZERO MEAN RANDOM PROCESS WITH AUTOCORRELATION $R_X(\tau)$. Let $Y(t) = f(t)X(t)$ WHERE $f(t)$ IS A GIVEN DETERMINISTIC FUNCTION.

(a) Compute the mean of $Y(t)$

(b) Compute the autocorrelation $R_Y(t_1, t_2)$

(c) Is $Y(t)$ stationary in the wide sense? Why or why not?

4. Both $\mathbf{X}$ & $\mathbf{Y}$ have pdf's $e^{-x}U(x)$. Form the transformation:

$$\mathbf{U} = 2\mathbf{X} + \mathbf{Y} \quad ; \quad \mathbf{V} = \mathbf{X} + \mathbf{Y}$$

(a) Compute the joint density function $f_{\mathbf{UV}}(u,v)$

(b) Sketch the u-v plane and clearly specify the region(s) over which $f_{\mathbf{UV}}(u,v)$ is not identically zero.

5. Let $\bar{x} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{X}_n$ where the $\mathbf{X}_n$'s are independent Cauchy random variables with pdf $f_{\mathbf{X}}(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}$.

(a) Compute the density of $\bar{x}$. Recall $\Phi_{\mathbf{X}}(\omega) = e^{-\alpha|\omega|}$.

(b) Comment on the applicability of the central limit theorem to this problem.

In this case the basic relationships (10-14) and (10-15) take the form

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\cos\omega\tau\,d\tau \qquad R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\cos\omega\tau\,d\omega \tag{10-17}$$

The *cross-power spectrum* $S_{xy}(\omega)$ of two processes $\mathrm{x}(t)$ and $\mathrm{y}(t)$ is the Fourier transform of their cross-correlation:

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)e^{-j\omega\tau}\,d\tau = S_{yx}^{*}(\omega) \tag{10-18}$$

The Fourier inversion formula gives

$$R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)e^{j\omega\tau}\,d\omega \tag{10-19}$$

and with $\tau = 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\,d\omega = R_{xy}(0) = E\{\mathrm{x}(t)\mathrm{y}^{*}(t)\} \tag{10-20}$$

If $\mathrm{x}(t)$ is the voltage across a two-terminal device and $\mathrm{y}(t)$ is the resulting input current, then the above equals the expected value of the power delivered to this device.

If the processes $\mathrm{x}(t)$ and $\mathrm{y}(t)$ are *orthogonal* (see page 298), then

$$R_{xy}(\tau) = 0 \qquad S_{xy}(\omega) = 0$$

In this case [see (10-7)]

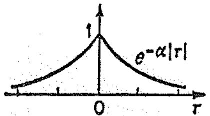
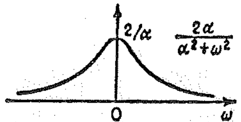
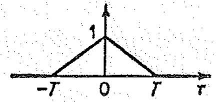
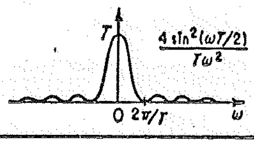
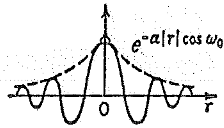
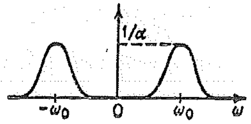
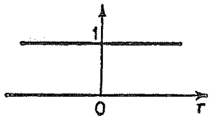
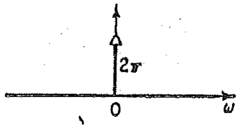
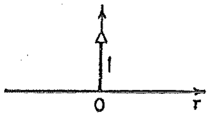
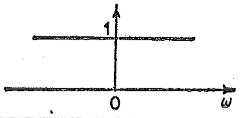
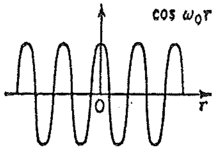
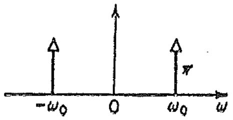
$$R_{x+y}(\tau) = R_x(\tau) + R_y(\tau) \qquad S_{x+y}(\omega) = S_x(\omega) + S_y(\omega)$$

Table 10-1 shows the correspondence between a process $\mathrm{x}(t)$, its autocorrelation $R(\tau)$, and power spectrum $S(\omega)$. The justification follows easily from definition (10-14) and the elementary properties of Fourier transforms [see also (9-85)].

Table 10-1

$\mathrm{x}(t)$	$R(\tau)$	$S(\omega)$
$a\mathrm{x}(t)$	$\lvert a\rvert^2R(\tau)$	$\lvert a\rvert^2S(\omega)$
$\dfrac{d\mathrm{x}(t)}{dt}$	$-\dfrac{d^2R(\tau)}{d\tau^2}$	$\omega^2S(\omega)$
$\dfrac{d^n\mathrm{x}(t)}{dt^n}$	$(-1)^n\dfrac{d^{2n}R(\tau)}{d\tau^{2n}}$	$\omega^{2n}S(\omega)$
$\mathrm{x}(t)e^{\pm j\omega_0t}$	$R(\tau)e^{\pm j\omega_0\tau}$	$S(\omega \mp \omega_0)$
See Sec. 10-6	$R(\tau)\cos\omega_0\tau$	$\frac{1}{2}[S(\omega + \omega_0) + S(\omega - \omega_0)]$

Table 10-2

$R(\tau)$	$S(\omega)$
	
	
	
	
	
	

In Table 10-2 we show a number of autocorrelations and their transforms. We leave the easy proofs as exercises.

Comment. The power spectrum $S(\omega)$ of a process $\mathrm{x}(t)$ can be expressed directly in terms of its second-order density $f(x_1,x_2;\tau)$. To this end we introduce the Fourier transform of $f(x_1,x_2;\tau)$ with respect to τ:

$$G(x_1,x_2;\omega) = \int_{-\infty}^{\infty} f(x_1,x_2;\tau)e^{-j\omega\tau}\,d\tau$$

Since [see (9-7)]

$$R(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2f(x_1,x_2;\tau)\,dx_1\,dx_2$$

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NAME ______________

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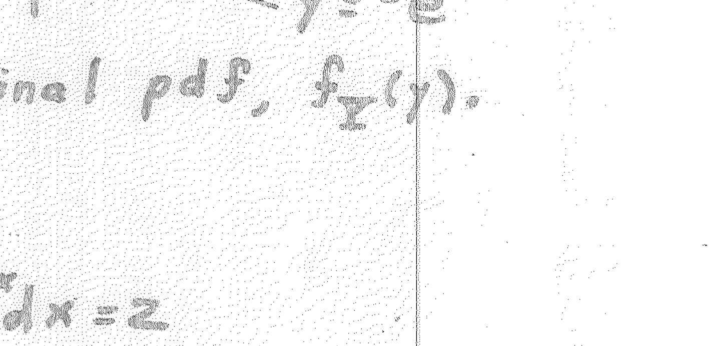
1. A joint pdf has uniform mass over the shown shaded area:



(a) What is the uniform height of $f_{XY}(x,y)$?

(b) Compute the marginal pdf, $f_Y(y)$.

Solution (a) $AREA = 2\int_0^\infty e^{-x}dx = 2$

$VOLUME = 1 \Longrightarrow HEIGHT = \frac{1}{2}$

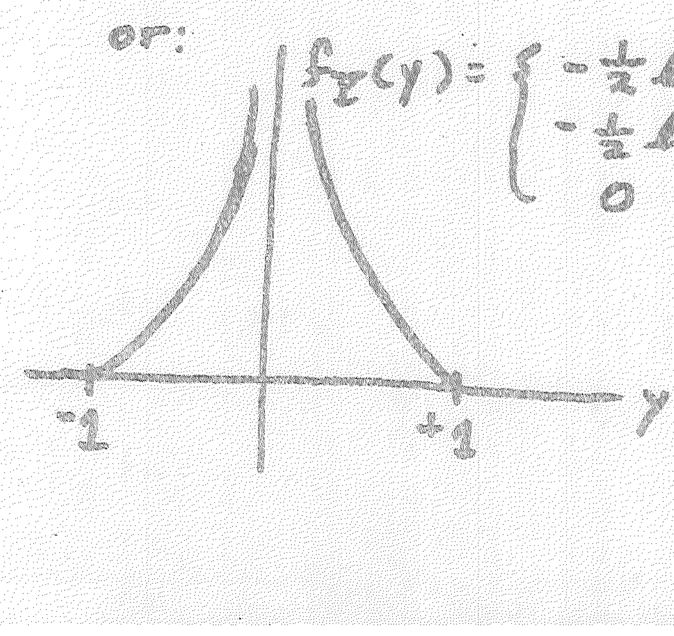
(b) $f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx$

obviously: $f_Y(y) = 0$ for $|y| > 1$

otherwise:

$$f_Y(y) = \begin{cases} \int_{x=0}^{-\ln y} \left(\frac{1}{2}\right) dx & ; 0 \le y \le 1 \\ \int_{x=0}^{-\ln(-y)} \left(\frac{1}{2}\right) dx & ; -1 \le y \le 0 \end{cases}$$

or:



$$f_Y(y) = \begin{cases} -\frac{1}{2}\ln y & ; 0 \le y \le 1 \\ -\frac{1}{2}\ln(-y) & ; -1 \le y < 0 \\ 0 & ; \text{otherwise} \end{cases}$$

$$= \begin{cases} -\frac{1}{2}\ln|y| & ; |y| \le 1 \\ 0 & ; |y| > 1 \end{cases}$$

2. A sample Y is taken from a normal pdf with mean η_y and variance σ^2. Form the random process:

$$X(t) = Y \quad ; -\infty < t < \infty$$

(a) Compute the mean and autocorrelation of $X(t)$. Is the process stationary?

(b) $X(t)$ is the input to a linear time-invariant filter with impulse response $h(t) = e^{-\alpha|t|}$. Let $Z(t)$ denote the filter output. Compute the mean & autocorrelation of $Z(t)$.

Solution (a) $\eta(t) = E[X(t)] = E[Y] = \eta_y$

$R_x(\tau) = E[X(t)X(t+\tau)] = E[Y^2]$

Since $\sigma^2 = E[Y^2] - \eta_y^2$, $R_x(\tau) = \sigma^2 + \eta_y^2$

Yes, the process is stationary.

(b) $S_z(\omega) = |H(j\omega)|^2 S_x(\omega)$

$S_x(\omega) = 2\pi(\sigma^2 + \eta_y^2)\,\delta(\omega)$

$|H(j\omega)|^2 = \left|\frac{2\alpha}{\alpha^2+\omega^2}\right|^2 \Rightarrow S_z(\omega) = \left(\frac{2}{\alpha}\right)^2 2\pi(\sigma^2 + \eta_y^2)\,\delta(\omega)$

$\therefore R_z(\tau) = \frac{4(\sigma^2 + \eta_y^2)}{\alpha^2}$

$\eta_z = H(0)\,\eta_x = \frac{2}{\alpha}\eta_y$

3. LET $X(t)$ DENOTE A ZERO MEAN RANDOM PROCESS WITH AUTOCORRELATION $R_X(\tau)$. Let $Y(t) = f(t)X(t)$ WHERE $f(t)$ IS A GIVEN DETERMINISTIC FUNCTION.

(a) Compute the mean of $Y(t)$

(b) Compute the autocorrelation $R_Y(t_1, t_2)$

(c) Is $Y(t)$ stationary in the wide sense? Why or why not?

Solution

(a) $\eta_y(t) = E[Y(t)] = E[f(t)X(t)] = f(t)E[X(t)] = 0$

since $E[X(t)] = 0$

(b)
$$
\begin{aligned}
R_Y(t_1, t_2) &= E[Y(t_1)\, Y(t_2)] \\
&= E[f(t_1)X(t_1)\, f(t_2)X(t_2)] \\
&= f(t_1) f(t_2) E[X(t_1)\, X(t_2)] \\
&= f(t_1) f(t_2) R_X(t_1, t_2) \\
&= f(t_1) f(t_2) R_X(\tau) \quad ; \quad \tau = t_1 - t_2
\end{aligned}
$$

(c) No way. R_Y is <u>not</u> strictly a function of $\tau = t_1 - t_2$ (unless we have the trivial case $f(t) = \text{CONSTANT}$)

4. Both X & Y have pdf's $e^{-x}U(x)$. Form the transformation:

$$U = 2X + Y \quad ; \quad V = X + Y$$

(a) Compute the joint density function $f_{UV}(u,v)$

(b) Sketch the u-v plane and clearly specify the region(s) over which $f_{UV}(u,v)$ is not identically zero.

Solution

(a)

$$u = 2x + y \qquad v = x + y$$
$$x = u - v \qquad y = 2v - u$$

$$f_{XY}(x,y) = e^{-(x+y)}\,U(x)\,U(y)$$

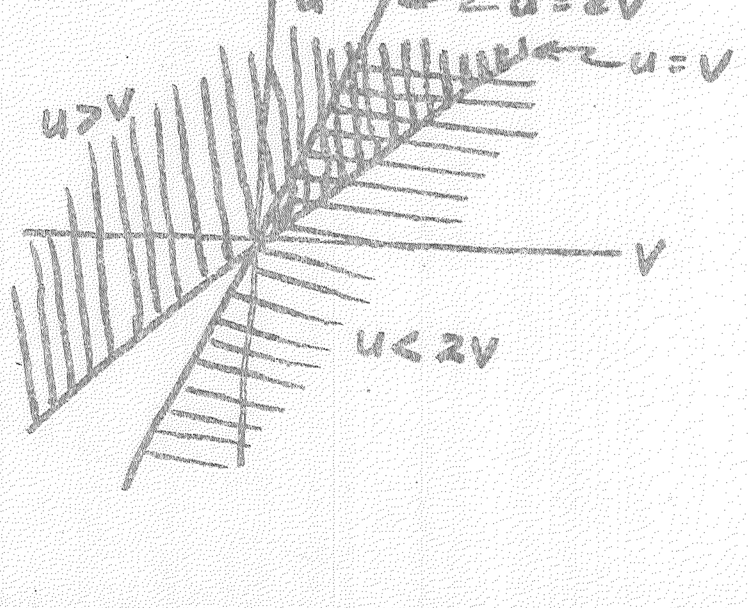
$$|J| = \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1$$

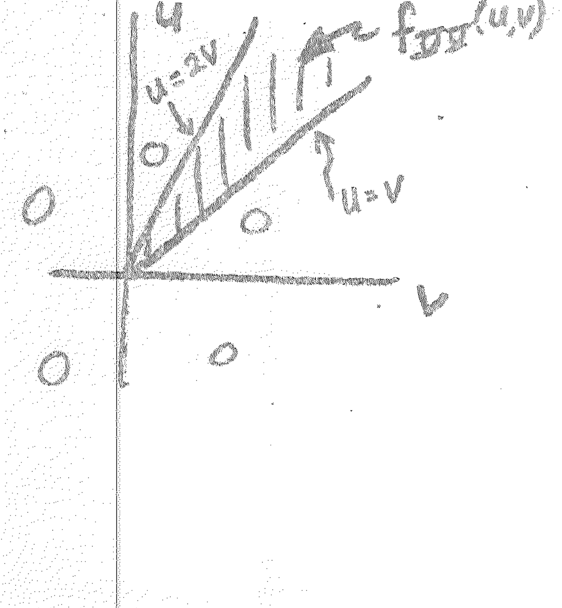
$$f_{UV}(u,v) = |J|\,e^{-(u-v+2v-u)}\,U(u-v)\,U(2v-u)$$
$$= e^{-v}\,U(u-v)\,U(2v-u)$$

(b)

$$U(u-v) = 1 \quad ; \quad u - v > 0$$
$$u > v$$

$$U(2v-u) = 1 \quad ; \quad 2v > u$$
$$u < 2v$$





5. Let $\overline{x} = \frac{1}{N}\sum_{n=1}^{N} X_n$ where the X_n's are independent Cauchy random variables with pdf $f_X(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}$.

(a) Compute the density of $\overline{x}$. Recall $\Phi_X(\omega) = e^{-\alpha|\omega|}$

(b) Comment on the applicability of the central limit theorem to this problem.

SOLUTION

(a) $\Phi_{\overline{x}}(\omega) = E[e^{j\omega\overline{x}}] = E[e^{j\omega\frac{1}{N}\sum_{n=1}^{N} x_n}]$

$= E[\prod_{n=1}^{N} e^{j(\frac{\omega}{N})x_n}]$

$= \prod_{n=1}^{N} E[e^{j(\frac{\omega}{N})x_n}]$ (since x_n's independent)

$= [E(e^{j\frac{\omega}{N}x})]^N$ (since identically distributed)

$= \Phi_X^N(\frac{\omega}{N}) = [e^{-\alpha|\frac{\omega}{N}|}]^N$

$= e^{-\alpha|\omega|} \Rightarrow f_{\overline{x}}(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}$

(b) The distribution of $\overline{x}$ is Cauchy, irrespective of the value of N. The central limit theorem is thus not applicable here. Note the Cauchy distribution has an infinite second moment so an assumption of proof of the central limit theorem is not fulfilled

EE505
Midterm #1
Summer, '82

Name ______________________________

1. We roll three dice: one red, one blue and one yellow. Consider the following events:

A = the red die shows 6
B = the sum of the red and yellow dice is three
C = the sum on all three dice is seven
D = the blue die turns up odd
E = the red die turns up odd
F = the sum of the red and yellow dice is eight

Classify each of the following groups of events as:

Independent (I)
Mutually Exclusive (M)
Neither (N)

No penalty for guessing.

A & B ____________

A & D ____________

A & E ____________

A & F ____________

B & C ____________

B & D ____________

B & E ____________

C & F ____________

D & E ____________

B, [redacted] F ____________

For the pdf:

$$f_X(x) = \begin{cases} Cx^3 & ; \ 0 \leq x \leq 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

find:
(a) the constant C
(b) $\overline{x}$
(c) var(X)

John has 6 red and 2 blue sox.
Frank has 2 red and 6 blue sox.
We choose a man at random and take a sock from him at random.
The sock is red.
What is the probability that the man we chose was John?

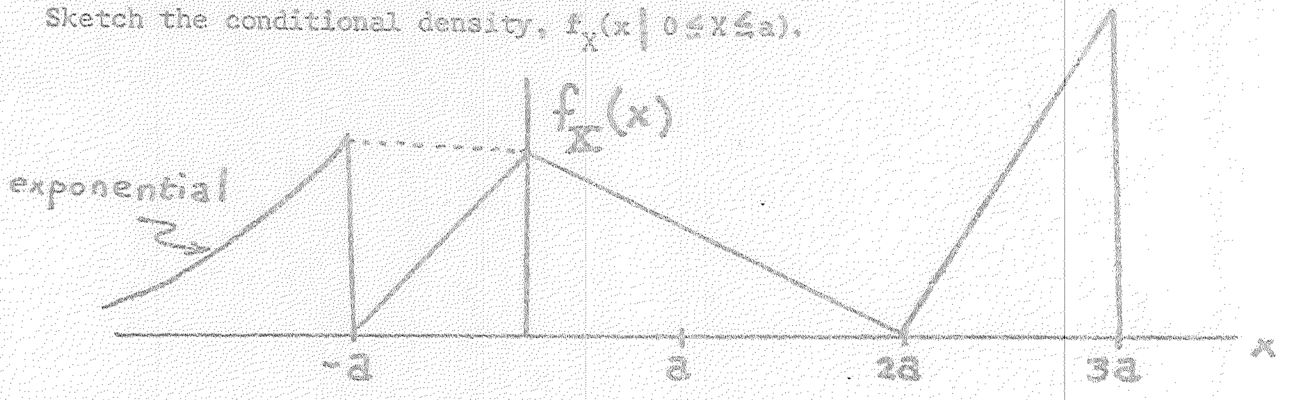
-4-

Let r be an integer. For the gamma distribution, if $b + 1 = r/2$ and $c=1/2$, the resulting random variable is called χ^2_r (chi-squared with r degrees of freedom.) Let X come from a distribution with $r = 4$. Compute

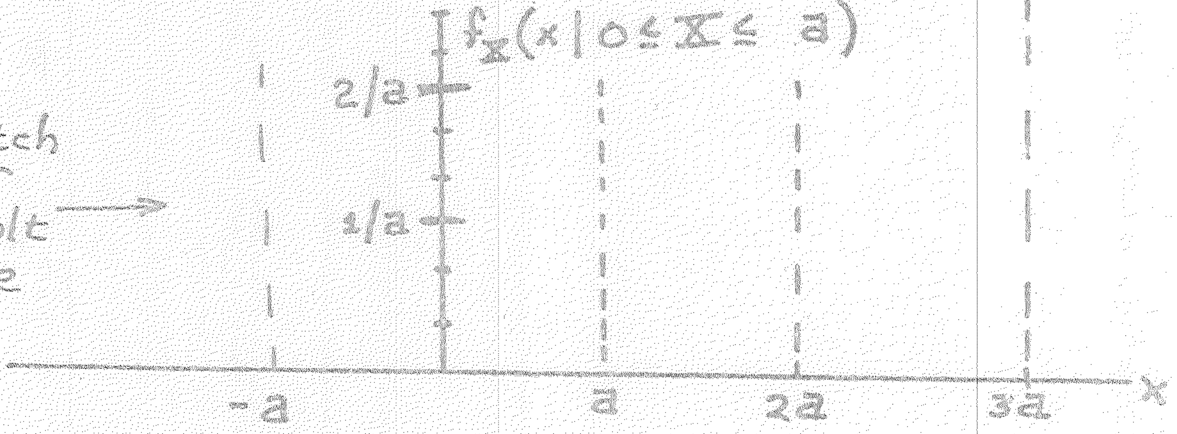
$$Pr(0 \leq X \leq 1)$$

Below is pictured a pdf.

Sketch the conditional density, $f_X(x \mid 0 \le X \le a)$.



sketch your result here



-6-

The Weibull distribution is defined by:

$$F_X(x) = \left[1 - \exp\left\{-\left(\frac{x}{A}\right)^B\right\}\right]\mu(x)$$

where A is the "scale" and B the "shape" parameter. Perform the random variable transformation:

$$Y = X^N$$

Y turns out also to be a Weibull random variable with, say, parameters $\hat{A}$ and $\hat{B}$. Compute these parameters in terms of A, B and N.

EE505
Midterm #1
Summer, '82

SOLUTIONS!

1. We roll three dice: one red, one blue and one yellow. Consider the following events:

A = the red die shows 6
B = the sum of the red and yellow dice is three
C = the sum on all three dice is seven
D = the blue die turns up odd
E = the red die turns up odd
F = the sum of the red and yellow dice is eight

Classify each of the following groups of events as:

Independent (I)
Mutually Exclusive (M)
Neither (N)

No penalty for guessing.

A & B M
A & D I
A & E M
A & F N
B & C N
B & D I
B & E I
C & F M
D & E I
D, [redacted] & F M

$P[B] = 2/36$
$P[E] = 1/2$
$P[B \cdot E] = P[\text{sum is 3, first is odd}]$
$= 1/36 = P[B]P[E]$

For the pdf:

$$f_X(x) = \begin{cases} cx^3 & ; \ 0 \le x \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

find:
(a) the constant c
(b) $\overline{x}$
(c) var(X)

(a) $\int_0^1 cx^3 dx = 1 = \frac{c}{4} x^4 \Big|_0^1 = \frac{c}{4} \Rightarrow c = 4$

(b) $\overline{x} = 4 \times \int_0^1 x^4 dx = \frac{4}{5} x^5 \Big|_0^1 = \frac{16}{20} = \frac{4}{5}$

(c) $\overline{x^2} = 4 \int_0^1 x^5 dx = \frac{4}{6} x^6 \Big|_0^1 = \frac{2}{3} \equiv = 0.04167$

$\text{var } X = \overline{x^2} - \overline{x}^2$

$= \frac{1}{24} - \frac{1}{(20)^2}$

$= \frac{1}{24} - \frac{1}{400} = \frac{50-3}{1200}$

$= \frac{47}{1200} = 0.039167$

John has 6 red and 2 blue sox.
Frank has 2 red and 6 blue sox.
We choose a man at random and take a sock from him at random.
The sock is red.
What is the probability that the man we chose was John?

J = John, F = Frank

r = red sock, b = blue sock

Find $P_r[J/r]$

From Bayes:

$$P_r[J/r] = \frac{P_r[J]P_r[r/J]}{P_r(J)P_r[r/J] + P_r[F]P_r[r/F]}$$

$$= \frac{\frac{1}{2}\cdot\frac{6}{8}}{\frac{1}{2}\cdot\frac{6}{8} + \frac{1}{2}\cdot\frac{2}{8}}$$

$$= \frac{6}{8} = \frac{3}{4} = 0.75$$

Let r be an integer. For the gamma distribution, if b + 1 = r/2 and c=1/2, the resulting random variable is called χ^2_r (chi-squared with r degrees of freedom.) Let X come from a distribution with r = 4. Compute

Pr(0≤X≤1)

In general:

$$f_X(x) = \frac{c^{b+1}}{\Gamma(b+1)} x^b e^{-cx} \mu(x)$$

$$P = P_r[0 \le X \le 1] = \int_0^1 f_X(x)\,dx$$

$$c = \tfrac{1}{2}, \quad b+1 = \tfrac{r}{2} = 2 \Rightarrow b = 1$$

$$f_X(x) = \frac{(\frac{1}{2})^2}{\Gamma(2)} x e^{-x/2} \mu(x) = \tfrac{1}{4} x e^{-x/2} \mu(x)$$

$$P = \tfrac{1}{4} \int_0^1 x e^{-x/2} \mu(x)\,dx$$

Integration by parts:

$$u = x \qquad dv = \tfrac{1}{4} e^{-x/2} dx$$

$$du = dx \qquad v = -\tfrac{1}{2} e^{-x/2}$$

$$\Rightarrow P = -\tfrac{1}{2} x e^{-x/2}\Big|_0^1 + \tfrac{1}{2}\int_0^1 e^{-x/2} dx$$

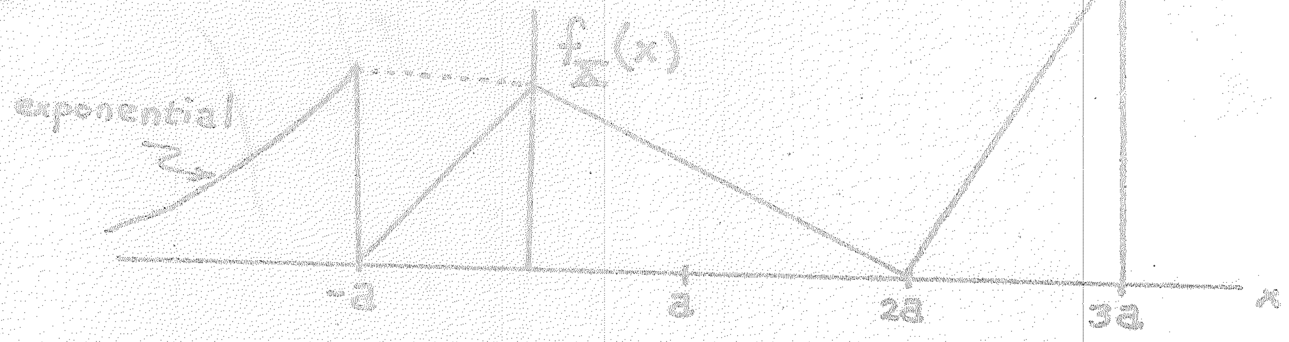
$$= -\tfrac{1}{2} e^{-1/2} + e^{-x/2}\Big|_1^0$$

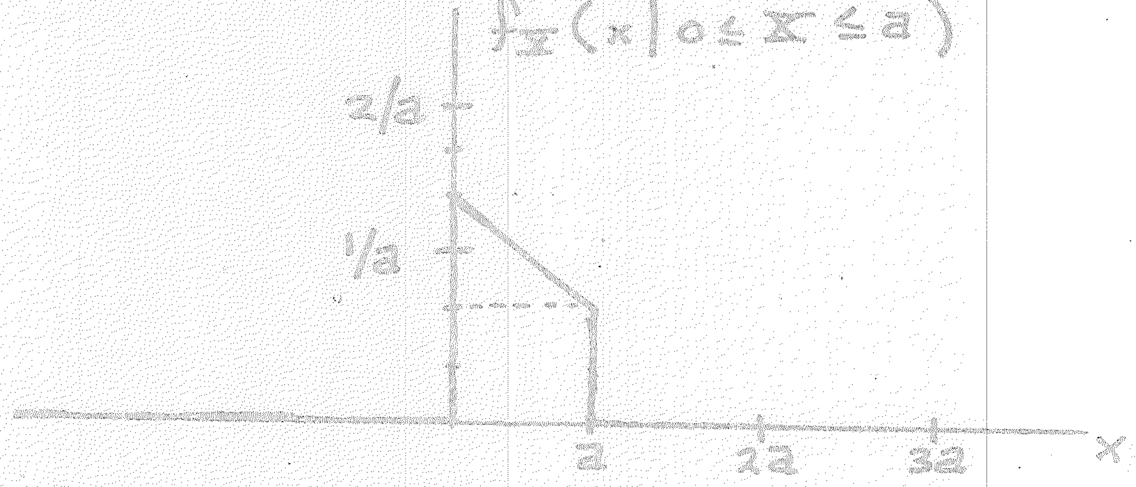
$$= -\tfrac{1}{2} e^{-\frac{1}{2}} + 1 - e^{-1/2}$$

$$= 1 - \tfrac{3}{2} e^{-1/2} \simeq 0.09020$$

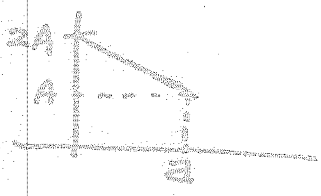
Below is pictured a pdf.

Sketch the conditional density, $f_X(x \mid 0 \le x \le a)$.





$f_X(x \mid 0 \le X \le a)$ looks like this $\Longrightarrow$ scaled to yield an area of one. In terms of A:



$$\text{area} = \frac{3}{2}Aa = 1 \Longrightarrow A = \frac{2}{3a}$$

Result sketched above

The. Weibull distribution is defined by:

$$F_X(x) = \left[1 - \exp\left\{\left(\frac{x}{A}\right)^B\right\}\right]\mu(x)$$

where A is the "scale" and B the "shape" parameter. Perform the random variable transformation:

$$Y = X^N$$

Y turns out also to be a Weibull random variable with, say, parameters $\hat{A}$ and $\hat{B}$. Compute these parameters in terms of A, B and N.

$$F_Y(y) = P_r[Y \le y] = P_r[X^N \le y]$$

over $0 \le x \le \infty$ $y = x^N$ is monotonic

$$\Longrightarrow x = y^{(1/N)}$$

Thus:

$$F_Y(y) = P_r\left[X \le y^{\frac{1}{N}}\right] = F_X\left(y^{\frac{1}{N}}\right)$$

$$= \left[1 - e^{-\left(\frac{y^{\frac{1}{N}}}{A}\right)^B}\right]\mu\left(y^{\frac{1}{N}}\right)$$

$$= \left[1 - e^{-\left(\frac{y}{A^N}\right)^{B/N}}\right]\mu(y)$$

$$= \left[1 - e^{-(y/\hat{A})^{\hat{B}}}\right]\mu(y)$$

Thus: $\hat{A} = A^N$

$\hat{B} = B/N$

505
MIDTERM #2
SUMMER '82

Name ______________

Score ______________ /150

1. Let X_n, $n = 1, 2, \ldots, N$ denote independent Cauchy random variables distributed thusly:

$$f_{X_n}(x_n) = \frac{\alpha/\pi}{\alpha^2 + x_n^2} \quad ; n = 1, 2, \ldots, N$$

Form the average: $\bar{X} = \frac{1}{N} \sum_{n=1}^{N} X_n$. Compute $f_{\bar{X}}(x)$.

HINTS		
	$\int y e^{-ay^2} dy = \frac{-1}{2a} e^{-ay^2}$	$\int_{-\infty}^{\infty} e^{-\alpha\lvert\omega\rvert} e^{j\omega x} d\omega = \frac{2\alpha}{\alpha^2 + x^2}$
	$\int x e^{-ax} dx = \frac{-e^{-ax}}{a}\left[x + \frac{1}{a}\right]$	$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$

-2-

N independent identically distributed random variables $\{ X_n \mid n=1,2,3,\ldots,N \}$ are zero mean and unit variance. We form the average:

$$\overline{x} = \frac{1}{N} \sum_{n=1}^{N} X_n$$

Find a lower bound on the probability that $\overline{x}$ lies between $-a$ and $a>0$.

-3-

X and Y are independent zero-mean normal random variables with variance σ^2. Let

$$Z = X/Y$$

Compute $f_Z(z)$.

-4-

Let X and Y denote <u>positive</u> random variables. i.e.

$$f_{XY}(x,y) = f_{XY}(x,y)\,\mu(x)\,\mu(y)$$

where $\mu(\cdot)$ denotes the unit step. Let

$$U = X^{Y} \quad , \quad V = X^{-2}$$

Find $f_{UV}(u,v)$. in terms of $f_{XY}(x,y)$.
<u>Include</u> <u>appropriate</u> <u>limits</u>

-5-

Consider three statistics: X_0, X_1 and X_2. Given:

X_1 and X_2 are orthogonal,
$E[X_0X_1] = E[X_0X_2] = 6$,
$E[X_1^2] = 2$, $E[X_2^2] = 3$

find the best linear estimate of X_0 in terms of X_1 and X_2.

Consider the conditional density:

$$f_Y(y/x) = \frac{1}{x} h(x) e^{-xy} \mu(y)$$

where $\mu(y)$ denotes the unit step.

(a) Compute the function $h(x)$.

(b) Statistics X and Y are taken from $f_{XY}(x,y)$: the joint density from which $f_Y(y/x)$ above was obtained. The value of X was 2. What is our corresponding m.s.e. of Y?
Give a number.

EE505
MIDTERM #2
SUMMER '82

Name ____________

Score ____________ /150

1. Let $\mathbb{X}_n$, $n = 1, 2, \ldots, N$ denote independent Cauchy random variables distributed thusly:

$$f_{\mathbb{X}_n}(x_n) = \frac{\alpha/\pi}{\alpha^2 + x_n^2} \quad ; n = 1, 2, \ldots, N$$

Form the average: $\bar{\mathbb{X}} = \frac{1}{N} \sum_{n=1}^{N} \mathbb{X}_n$. Compute $f_{\bar{\mathbb{X}}}(x)$.

From hint:

$$\Phi_{\mathbb{X}_n}(\omega) = e^{-\alpha|\omega|}$$

Thus:

$$\Phi_{\bar{\mathbb{X}}}(\omega) = E[e^{j\omega\bar{\mathbb{X}}}] \underset{\uparrow}{=} \Phi^N_{\mathbb{X}_n}\left(\frac{\omega}{N}\right)$$

showed in class for average.

$$= \left[e^{-\alpha\left|\frac{\omega}{N}\right|}\right]^N$$

$$= e^{-\alpha|\omega|}$$

same thing!

Hence: $\bar{\mathbb{X}}$ is Cauchy:

$$f_{\bar{\mathbb{X}}}(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}$$

Note: This was prob. 8-19 on p. 275 of text.

-2-

N independent identically distributed random variables $\{X_n \mid n=1,2,3,\ldots,N\}$ are zero mean and unit variance. We form the average:

$$\overline{x} = \frac{1}{N}\sum_{n=1}^{N} X_n$$

Find a lower bound on the probability that $\overline{x}$ lies between $-a$ and $a>0$.

$\overline{x}$ has mean $= 0$
and variance $= \frac{1}{N} = \sigma_{\overline{x}}^2$

From Chebychev:

$$P_r\left[\,|X| < k\sigma\right] \geq 1 - \frac{1}{k^2}$$

$$k\sigma = a \quad , \quad \overline{x} = X$$

$$P_r\left[\,|\overline{x}| < a\right] \geq 1 - \left(\frac{\sigma_{\overline{x}}}{a}\right)^2 = 1 - \frac{1}{(Na)^2}$$

Note: $P_r\left[\,|\overline{x}| < a\right] \xrightarrow[N\to\infty]{} 1$

X and Y are independent zero-mean normal random variables with variance σ^2. Let

$$Z = X/Y$$

Compute $f_Z(z)$.

From Class work:

$$f_Z(z) = \int_{-\infty}^{\infty} |y| \, f_{XY}(zy, y)\, dy$$

Since: $f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$

$$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} |y| \frac{1}{2\pi\sigma^2} e^{-\frac{z^2y^2+y^2}{2\sigma^2}} dy$$

$$= 2\int_0^{\infty} y \frac{1}{2\pi\sigma^2} e^{-\frac{y^2(z^2+1)}{2\sigma^2}} dy$$

$$= \frac{1}{\pi\sigma^2}\left(\frac{-\sigma^2}{z^2+1}\right) e^{-y^2(z^2+1)/2\sigma^2} \Big|_{y=0}^{\infty}$$

$$= \frac{1/\pi}{z^2+1} \Longleftarrow \text{Cauchy}$$

-4-

Let $\mathbf{X}$ and $\mathbf{Y}$ denote <u>positive</u> random variables. ie

$$f_{XY}(x,y) = f_{XY}(x,y)\,\mu(x)\,\mu(y)]$$

where $\mu(\cdot)$ denotes the unit step. Let

$$\mathbf{U} = \mathbf{X}^{\mathbf{Y}} \quad , \quad \mathbf{V} = \mathbf{X}^{-2}$$

Find $f_{UV}(u,v)$. in terms of $f_{XY}(x,y)$.
<u>Include appropriate limits</u>

$\mathbf{X} = \mathbf{V}^{-\frac{1}{2}}$

$\mathbf{U} = \left(\mathbf{V}^{-\frac{1}{2}}\right)^{\mathbf{Y}} = \mathbf{V}^{-\frac{1}{2}\mathbf{Y}} \Rightarrow \mathbf{Y} = -2\dfrac{\ln \mathbf{U}}{\ln \mathbf{V}} \quad , \quad \mathbf{X} = \mathbf{V}^{-\frac{1}{2}}$

$$\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} 0 & -\frac{1}{2}v^{-3/2} \\ \frac{-2}{u \ln v} & \frac{dy}{dv} \end{vmatrix} = \left|\frac{1}{v^{3/2}\, u \ln v}\right|$$

$$\Rightarrow f_{UV}(u,v) = \frac{1}{|v^{3/2} u \ln v|}\, f_{XY}\left(v^{-\frac{1}{2}}, -2\frac{\ln u}{\ln v}\right)$$

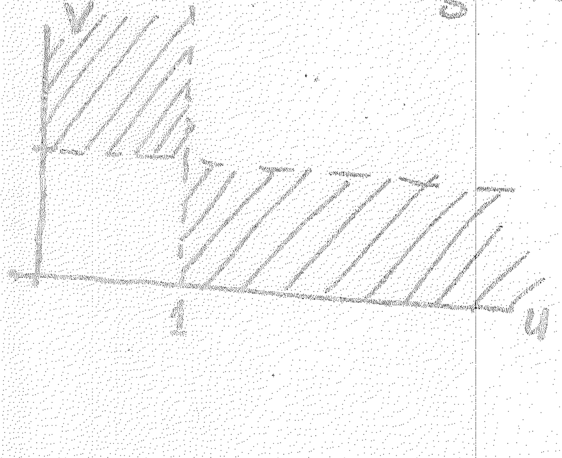
LIMITS: $\mathbf{X} > 0 \Rightarrow \mathbf{V}^{-\frac{1}{2}} > 0$

$\mathbf{Y} \geq 0 \Rightarrow 2\dfrac{\ln \mathbf{U}}{\ln \mathbf{V}} \leq 0$

ⓐ For $0 \leq \mathbf{U} \leq 1 \Rightarrow \mathbf{V} > 1$

For $\mathbf{U} > 1 \Rightarrow 0 < \mathbf{V} < 1$

From transformation, it is obvious that both $\mathbf{U}$ & $\mathbf{V}$ must be positive. Convergence is in shaded region:



Consider three statistics: X_0, X_1 and X_2. Given:

X_1 and X_2 are orthogonal,

$E[X_0 X_1] = E[X_0 X_2] = 6$,

$E[X_1^2] = 2$, $E[X_2^2] = 3$

find the best linear estimate of X_0 in terms of X_1 and X_2.

$X_0 = a_1 X_1 + a_2 X_2$

where a_1 & a_2 satisfy

$R_{01} = a_1 R_{11} + a_2 R_{12}$

$R_{02} = a_1 R_{12} + a_2 R_{22}$

OR: $\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

But: $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

hence, linear m.s.e. is

$\hat{X}_0 = 3X_1 + 2X_2$

-6-

Consider the conditional density:

$$f_Y(y/x) = \frac{1}{x} h(x) e^{-xy} \mu(y)$$

where $\mu(y)$ denotes the unit step.

(a) Compute the function $h(x)$.

(b) Statistics X and Y are taken from $f_{XY}(x,y)$: the joint density from which $f_Y(y/x)$ above was obtained. The value of X was 2. What is our corresponding m.s.e. of Y?
Give a number.

(a) for all x:

$$\int_{-\infty}^{\infty} f_Y(y|x)dy = 1 = \frac{1}{x} h(x) \int_0^\infty e^{-xy} dx$$

$$= \frac{1}{x^2} h(x) \Rightarrow h(x) = x^2$$

(b) m.s.e. estimate is:

$$\hat{Y} = g(X) = E[Y/X]$$

$$E[Y/X] = \int_{-\infty}^{\infty} y f_Y(y/x) dy$$

$$= x \int_0^\infty y e^{-xy} dy$$

$$u = yx \quad dv = e^{-xy} dy$$

$$du = x dy \quad v = -\frac{1}{x} e^{-xy}$$

$$\Rightarrow E[Y/X] = -ye^{-xy}\Big|_{y=0}^{\infty} + \int_0^\infty e^{-xy} dy = \frac{1}{x}$$

$$\text{for } X = 2 \Rightarrow \hat{Y} = \frac{1}{2}$$

EE505 FINAL EXAM
8/19/82

NAME ______________________

TEST GRADE ______________ /20

1. Given the characteristic function:

$$\Phi_X(\omega) = \frac{A}{\cosh a\omega}$$

where "a" is a parameter, compute:

(a) A (b) $E[X]$ (c) $\mathrm{var}(X)$

-2-

The "lognormal density" is so named because it is the density of X if $\ln X$ is normally distributed with, say, mean η and variance σ^2. Compute the density function of a lognormal random variable. Remember limits.

-3-

The spectral density of zero mean white noise is:

$$S_{\mathbb{X}}(\omega) = \text{Const}; \text{ all } \omega$$

Let $\mathbb{X}(t)$ denote such a process with const. = 1.

We pass $\mathbb{X}(t)$ thru a filter with transfer function:

$$H(\omega) = \begin{cases} j\cos\omega & ; |\omega| \leq \frac{\pi}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

Compute the output signal noise level

-4-

Let A & Θ denote independent random variables. A has mean "a" and variance σ_a^2. Θ has characteristic function $\Phi_\Theta(\omega)$. Form the random amplitude-phase rotating phasor:

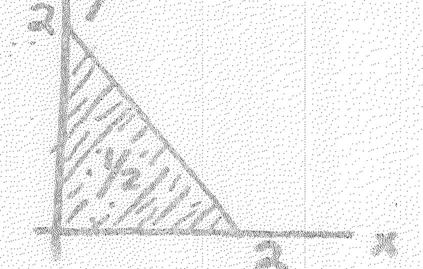
$$X(t) = A e^{j[2\pi f t + \Theta]} \quad ; \; f = \text{given constant}$$

(a) Compute the mean and autocorrelation of $X(t)$ in terms of a, σ_a^2 and $\Phi_\Theta(\omega)$

(b) Is the process stationary?

-5-

Consider the joint density:



$$f_{XY} = \begin{cases} \frac{1}{2} \ ; \text{shaded area} \\ 0 \ ; \text{otherwise} \end{cases}$$

The corresponding radial distance is:

$$R = \sqrt{X^2 + Y^2}$$

Compute:

$$E[R^2] = \int_0^{2\sqrt{2}} f_R(r)\, r^2 dr$$

-6-

Let $\{X_n \mid n=1,2,\ldots,N>>1\}$ denote identically distributed independent random variables with unknown mean η and known variance σ^2. We form the average:

$$\overline{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$$

Assume N is sufficiently large for application of the central limit theorem.

(a) What is the probability that $\overline{X}$ lies within α standard deviations of η?

(b) How many samples do we need to be assured that there is about a 99% chance $\overline{X}$ is within $\frac{1}{100}$th of a standard deviation from η?

- 7 -

Given the stochastic differential equation:

$$\sum_{n=0}^{N} a_n Y^{(n)}(t) = X(t)\ ;\ Y^{(n)}(0)=0\ ,\ n=1,2,\ldots,N-1$$

we compute the cross correlation by solving the deterministic differential equation:

$$\sum_{n=0}^{N} a_n \left(\frac{\delta}{\delta t_2}\right)^n R_{XY}(t_1,t_2) = R_X(t_1,t_2)$$

with initial conditions: $\left(\frac{\delta}{\delta t_2}\right)^n R_{XY}(t_1,0)=0$; $n=0,1,\ldots,N-1$. <u>DERIVE</u> the differential equation we must solve to find $R_Y(t_1,t_2)$. Don't forget the initial conditions.

– 8 –

Let $X(t)$ denote a stochastic process with mean η_x and autocorrelation $R_X(\tau)$. $X(t)$ is ergodic in the mean. Let A denote a random variable with mean a and variance σ_a^2. A & $X(t)$ are uncorrelated. Define:

$$Y(t) = A + X(t)$$

Is $Y(t)$ ergodic in the mean? Show your work.

EE505

SOLUTIONS to Final Examination; 8/19/82

1. Given the characteristic function:

$$\Phi_X(\omega) = \frac{A}{\cosh a\omega}$$

where "a" is a parameter, compute:

(a) A (b) $E[X]$ (c) $\mathrm{var}(X)$

(a) $\Phi_X(0) = 1 \Rightarrow A = 1$

(b) $jE[X] = \frac{d}{d\omega}\Phi_X(0) = \frac{d}{d\omega}\frac{1}{\cosh a\omega}\Big|_{\omega=0}$

$$= \frac{-a\sinh a\omega}{\cosh^2 a\omega}\Big|_{\omega=0} = 0$$

$$\Rightarrow E(X) = 0$$

(c) $\mathrm{var}\, X = E[X^2]$;

$$j^2 E[X^2] = \frac{d}{d\omega}\,\frac{-a\sinh a\omega}{\cosh^2 a\omega}$$

$$= -a\left[\frac{\cosh^2 a\omega \; a\cosh a\omega - 2a\sinh^2 a\omega\cosh a\omega}{\cosh^4 a\omega}\right]_{\omega=}$$

$$= -a\left[a\right] = -a^2$$

$$\Rightarrow \mathrm{var}\, X = a^2$$

The "lognormal density" is so named because it is the density of X if $\ln X$ is normally distributed with, say, mean η and variance σ^2. Compute the density function of a lognormal random variable. Remember limits.

$$Y = \ln X \sim \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(y-\eta)^2/2\sigma^2}$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\Rightarrow f_X(x) = \frac{1}{x\sqrt{2\pi}\,\sigma} e^{-(\ln x-\eta)^2/2\sigma^2} \mu(x)$$

The spectral density of zero mean white noise is:

$$S_X(\omega) = \text{Const}; \text{ all } \omega$$

Let $X(t)$ denote such a process with const. = 1.

We pass $X(t)$ thru a filter with transfer function:

$$H(\omega) = \begin{cases} j\cos\omega & ; |\omega| \le \frac{\pi}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

Compute the output signal noise level

$$S_Y(\omega) = S_X(\omega)|H(\omega)|^2$$

$$= \begin{cases} \cos^2\omega & ; \omega \le \frac{\pi}{2} \\ 0 & ; \omega > \frac{\pi}{2} \end{cases}$$

$$E[Y^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1 + \cos 2\omega]\frac{1}{2}d\omega$$

$$= \frac{1}{4\pi}\left[\omega - \frac{1}{2}\sin 2\omega\right]\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{4\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{4}$$

-4-

Let A & Θ denote independent random variables. A has mean "a" and variance σ_a^2. Θ has characteristic function $\Phi_\Theta(\omega)$. Form the random amplitude-phase rotating phasor:

$$X(t) = A e^{j[2\pi f t + \Theta]} \quad ; \; f = \text{given constant}$$

(a) Compute the mean and autocorrelation of $X(t)$ in terms of a, σ_a^2 and $\Phi_\Theta(\omega)$

(b) Is the process stationary?

(a) $\eta_x(t) = E[X(t)] = E[A e^{j(2\pi f t + \Theta)}]$

$= E[A]\, E[e^{j(2\pi f t + \Theta)}] \longleftarrow$ follows from independence

$= a e^{j2\pi f t} E[e^{j\Theta}]$

Since $\Phi_\Theta(\omega) = E[e^{j\omega\Theta}]$;

$\eta_x(t) = a \Phi_\Theta(1) e^{j2\pi f t}$

$R_X(t_1, t_2) = E[A^2 e^{j[2\pi f(t_1 - t_2) + 2\Theta]}]$

$= [\sigma_a^2 + a^2]\, e^{j2\pi f(t_1 - t_2)}\, E[e^{j2\Theta}]$

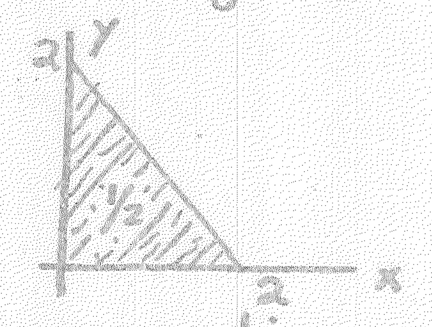
$\Rightarrow R(t_1 - t_2) = R(\tau) = (\sigma_a^2 + a^2)\, \Phi_\Theta(2)\, e^{j2\pi f\tau}$

(b) Mean changes with t

$\Longrightarrow$ not stationary

-5-

Consider the joint density:



$$f_{XY} = \begin{cases} \frac{1}{2} & ;\text{shaded area} \\ 0 & ;\text{otherwise} \end{cases}$$

The corresponding radial distance is:

$$R = \sqrt{X^2 + Y^2}$$

Compute:

$$E[R^2] = \int_0^{2\sqrt{2}} f_R(r)\, r^2 dr$$

$$E[R^2] = \int_0^{2\sqrt{2}} r^2 f_R(r) dr$$

$$= \iint (x^2 + y^2) f_{XY}(x,y) dx dy$$

$$= \frac{1}{2} \int_{y=0}^{2} \int_{x=0}^{2-y} (x^2 + y^2) dx \; dy$$

$$= \frac{1}{2} \int_{y=0}^{2} \frac{1}{3} x^3 + y^2 x \Big|_0^{2-y} dy$$

$$= \frac{1}{2} \int_0^2 \left[\frac{1}{3}(2-y)^3 + y^2(2-y) \right] dy$$

$$= \frac{1}{2} \left[\frac{1}{3} \frac{(2-y)^4}{(-4)} + \frac{2}{3} y^3 - \frac{1}{4} y^4 \right]_0^2$$

$$= \frac{1}{2} \left[\frac{1}{3} \left(\frac{2^4}{4} \right) + \frac{2}{3} 2^3 - \frac{1}{4} 2^4 \right]$$

$$= \frac{1}{2} \left[\frac{16}{12} + \frac{16}{3} - \frac{16}{4} \right] =$$

$$= \frac{1}{2} \left[\frac{16 + 64 - 48}{12} \right]$$

$$= 4/3$$

-6-

Let $\{X_n \mid n=1,2,\dots,N>>1\}$ denote identically distributed independent random variables with unknown mean η and known variance σ^2. We form the average:

$$\overline{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$$

Assume N is sufficiently large for application of the central limit theorem.

(a) What is the probability that $\overline{X}$ lies within α standard deviations of η?

(b) How many samples do we need to be assured that there is about a 99% chance $\overline{X}$ is within $\frac{1}{100}$th of a standard deviation from η?

From central limit theorem $\overline{X}$ is approximately distributed normal with mean η and variance σ^2/N.

(a) We want to find:

$$p = \Pr[\eta - \alpha\sigma \le \overline{X} \le \eta + \alpha\sigma]$$

$$= \Pr\left[-\alpha\sqrt{N} < \frac{\overline{X}-\eta}{\sigma/\sqrt{N}} = Z < \alpha\sqrt{N}\right]$$

$$= 2\Pr[0 \le Z \le \alpha\sqrt{N}]$$

$$= 2 \operatorname{erf} \alpha\sqrt{N}$$

(b) $P = 0.99$, $\alpha = \frac{1}{100}$, $N = ?$

$$0.495 = \operatorname{erf} \frac{\sqrt{N}}{100}$$

from erf table: $\frac{\sqrt{N}}{100} \simeq 2.60$

$\sqrt{N} \simeq 260$

$N \simeq (260)^2$

$\simeq 67,600$ samples

-7-

Given the stochastic differential equation:

$$\sum_{n=0}^{N} a_n \, Y^{(n)}(t) = X(t) \; ; \; Y^{(n)}(0) = 0 , \; n = 0,1,2,\ldots,N-1$$

we compute the cross correlation by solving the deterministic differential equation:

$$\sum_{n=0}^{N} a_n \left(\frac{\delta}{\delta t_2}\right)^n R_{XY}(t_1,t_2) = R_X(t_1,t_2)$$

with initial conditions: $\left(\frac{\delta}{\delta t_2}\right)^n R_{XY}(t_1,0)=0$; $n=0,1,\ldots,N-1$. **DERIVE** the differential equation we must solve to find $R_Y(t_1,t_2)$. Don't forget the initial conditions.

From top equation:

$$\sum_{n=0}^{N} a_n \, Y^{(n)}(t_1) \, Y(t_2) = X(t_1) Y(t_2) \qquad (*)$$

Since:

$$R_Y(t_1,t_2) = E\left[Y(t_1)\,Y(t_2)\right]$$

It follows that:

$$\left(\frac{\delta}{\delta t_1}\right)^n R_Y(t_1,t_2) = E\left[Y^{(n)}(t_1)\,Y(t_2)\right]$$

Taking $E(\cdot)$ of both sides of $(*)$ gives:

$$\sum_{n=0}^{N} a_n \left(\frac{\delta}{\delta t_1}\right)^n R_Y(t_1,t_2) = R_{XY}(t_1,t_2)$$

To Get Initial conditions, note:

$$\left(\frac{\delta}{\delta t_1}\right)^n R_Y(0,t_2) = E\left[Y^{(n)}(0)\;Y(t_2)\right]$$

But $Y^{(n)}(0)=0$ and the initial conditions are:

$$\left(\frac{\delta}{\delta t_1}\right)^n R_Y(0,t_2) = 0 \; ; \quad n = 0,1,\ldots,N-1$$

\- 8 -

Let $X(t)$ denote a stochastic process with mean η_x and autocorrelation $R_X(\tau)$. $X(t)$ is ergodic in the mean. Let A denote a random variable with mean a and variance σ_a^2. A & $X(t)$ are uncorrelated. Define:

$$Y(t) = A + X(t)$$

Is $Y(t)$ ergodic in the mean? Show your work.

$$\eta_T = \langle Y \rangle = \frac{1}{2T}\int_{-T}^{T}[X(t)+A]\,dt$$

$$\Longrightarrow E\langle Y \rangle = \frac{1}{2T}\int_{-T}^{T}[\eta_x + a]\,dt = \eta_x + a$$

This is same as:

$$E[Y(t)] = E[X(t)+A] = \eta_x + a = E\langle Y \rangle$$

Criterion # 1 for ergodicity checks.

Second is: Does $\lim_{T\to\infty} \sigma^2_{\eta_T} = 0$ where:

$$\sigma^2_{\eta_T} = \frac{1}{T}\int_0^{2T}\left(1-\frac{\tau}{2T}\right)\left(R_Y(\tau) - \eta_Y^2\right)d\tau$$

Since $\eta_y = \eta_x + a$, and

$$R_Y(\tau) = E[(X(t)+A)(X(t+\tau)+A)]$$

$$= R_X(\tau) + 2a\eta_x + (\sigma_a^2 + a^2)$$, we have:

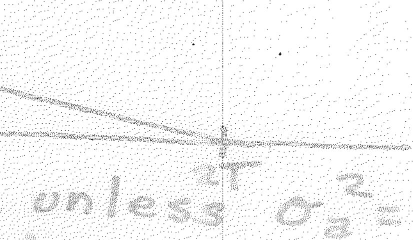
$$\sigma^2_{\eta_T} = \frac{1}{T}\int_0^{2T}\left(1-\frac{\tau}{2T}\right)\left[R_X(\tau) + \cancel{2a\eta_x} + (\sigma_a^2 + \cancel{a^2}) - \eta_x^2 - \cancel{a^2} - \cancel{2a\eta_x}\right]$$

Since X is ergodic we know:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^{2T}\left(1-\frac{\tau}{2T}\right)\left(R_X(\tau) - \eta_x^2\right) = 0$$

Thus:

$$\lim_{T\to\infty}\sigma^2_{\eta_T} = \lim_{T\to\infty}\frac{1}{T}\int_0^{2T}\left(1-\frac{\tau}{2T}\right)\sigma_a^2\,d\tau = \sigma_a^2$$



Thus, Y is <u>not</u> ergodic unless $\sigma_a^2 = 0$

(ie $A = a$

EE505, Summer '83

Test Problem:

Consider the joint pdf $(a>0,\ b>0,\ c>0)$:

$$f_{XY}(x,y) = A\,y^c\,e^{-ay}\exp(-bxy)\,u(x)\,u(y)$$

(a) Compute A.

(b,c) Find the marginal densities:

$$f_X(x) \text{ and } f_Y(y)$$

(include limits)

(d) Under what condition(s) are the moments

$$E[X^n\ Y^m] = m_{n,m} \text{ not finite.}$$

(e) Find $E[Y^m] = m_{0,m}$

(f) With $d>0$, define

$$Z = \frac{Y+d}{X}$$

Find $F_Z(z)$ when $c=1$

I, ______________________
(print your name)

have neither given assistance nor received help on this exam. The work is mine alone. Any reference source is noted in my work.

X______________________ ______________
sign date signed

(Hand in this sheet with your work)

Due in class@9:40 A.M on Wed, Aug 10, 1983.

Solutions: EE505 Summer '83

(a) $1 = \iint_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$

$= A\int_{y=0}^{\infty} y^c e^{-ay} \int_0^{\infty} e^{-bxy}\,dx\,dy$

$= A\int_{y=0}^{\infty} y^c e^{-ay} \frac{1}{by}\,dy$

$= A\frac{1}{b}\int_{y=0}^{\infty} y^{c-1} e^{-ay}\,dy$

Let $\xi = ay \Rightarrow y = \xi/a$

$1 = \frac{A}{b}\int_{\xi=0}^{\infty} \left(\frac{\xi}{a}\right)^{c-1} e^{-\xi}\,\frac{d\xi}{a}$

$= \frac{A}{b\,a^c}\int_0^{\infty} \xi^{c-1} e^{-\xi}\,d\xi = \frac{A}{ba^c}\Gamma(c)$

$\Longrightarrow A = ba^c/\Gamma(c)$

(b) $f_{XY}(x,y) = \frac{ba^c}{\Gamma(c)} y^c e^{-ay}\; e^{-bxy}\; u(x)u(y)$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

$= \frac{ba^c}{\Gamma(c)} y^c e^{-ay}\left[\int_{x=0}^{\infty} e^{-bxy}\,dx\right]u(y)$

$= \frac{ba^c}{\Gamma(c)} y^c e^{-ay}\left[\frac{1}{by}\right]u(y)$

$= \frac{a^c}{\Gamma(c)} y^{c-1} e^{-ay}\; u(y)$

(c) $f_{\mathbf{X}}(x) = \frac{ba^c}{\Gamma(c)} \int_{y=0}^{\infty} y^c e^{-ay} e^{-bxy} dy$

$$= \frac{ba^c}{\Gamma(c)} \int_{y=0}^{\infty} y^c e^{-(a+bx)y} dy$$

Let $\xi = (a+bx)y$

$$f_{\mathbf{X}}(x) = \frac{ba^c}{\Gamma(c)} \int_{\xi=0}^{\infty} \left(\frac{\xi}{a+bx}\right)^c e^{-\xi} \frac{d\xi}{a+bx} \, u(x)$$

$$= \frac{ba^c}{\Gamma(c)(a+bx)^c} \int_{\xi=0}^{\infty} \xi^c e^{-\xi} d\xi \, u(x)$$

$$= \frac{ba^c \Gamma(c+1)}{\Gamma(c)(a+bx)^{c+1}} u(x)$$

$$= \frac{ba^c \, c}{(a+bx)^{c+1}} \, u(x)$$

(d) $m_{nm} = \frac{ba^c}{\Gamma(c)} \int_{x=0}^{\infty} x^n \int_{y=0}^{\infty} y^{c+m} e^{-ay} e^{-bxy} dy$

$$= \frac{ba^c}{\Gamma(c)} \int_{x=0}^{\infty} x^n \int_{y=0}^{\infty} y^{c+m} e^{-y(a+bx)} dy$$

Let $\xi = (a+bx)y$ in y integral:

$$m_{nm} = \frac{ba^c}{\Gamma(c)} \int_{x=0}^{\infty} x^n \int_{\xi=0}^{\infty} \left(\frac{\xi}{a+bx}\right)^{c+m} e^{-\xi} \frac{d\xi}{a+bx}$$

-3-

$$= \frac{ba^c}{\Gamma(c)} \int_{x=0}^{\infty} \frac{x^n}{(a+bx)^{c+m+1}} \int_{q=0}^{\infty} q^{c+m} e^{-\frac{q}{d}}\, q d$$

$$= ba^c \frac{\Gamma(c+m+1)}{\Gamma(c)} \int_0^{\infty} \frac{x^n\,dx}{(a+bx)^r} \quad ; r = c+m+1$$

The integrand is bounded. Divergence occurs only if from the tail. Consider

$$I \equiv \int_A^{\infty} \frac{x^n\,dx}{(a+bx)^r} \qquad \text{for } A >> 1$$

$$I \approx \int_A^{\infty} \frac{x^n\,dx}{b^r x^r} = b^{-r} \int_A^{\infty} x^{n-r}\,dx$$

$$= b^{-r} \frac{1}{n-r+1} x^{n-r+1} \Big|_A^{\infty}$$

$$\lim_{x\to\infty} x^{n-r+1} = \begin{cases} 0 & ; n-r+1 > 0 \\ \infty & ; n-r+1 < 0 \\ 1 & ; n-r+1 = 0 \end{cases}$$

Thus $I = \infty$ for $n-r+1 \geq 0$

or $n-c-m-1+1$

$= n-m-c \geq 0$

and $m_{nm} = \infty$ for $n-m-c \geq 0$

(e) From (b):

$$f_Y(y) = \frac{a^c}{\Gamma(c)} y^{c-1} e^{-ay} u(y)$$

$$E[Y^m] = \frac{a^c}{\Gamma(c)} \int_0^\infty y^{m+c-1} e^{-ay} dy$$

$$\xi = ay$$

$$E[Y^m] = \frac{a^c}{\Gamma(c)} \int_0^\infty \left(\frac{\xi}{a}\right)^{m+c-1} e^{-\xi} \frac{d\xi}{a}$$

$$= \frac{a^c}{\Gamma(c)\, a^{m+c}} \int_0^\infty \xi^{m+c-1} e^{-\xi} d\xi$$

$$= \frac{a^{-m}}{\Gamma(c)} \Gamma(m+c) \quad \left(\text{note } \frac{f_Y}{a} \text{ is gamma pdf}\right)$$

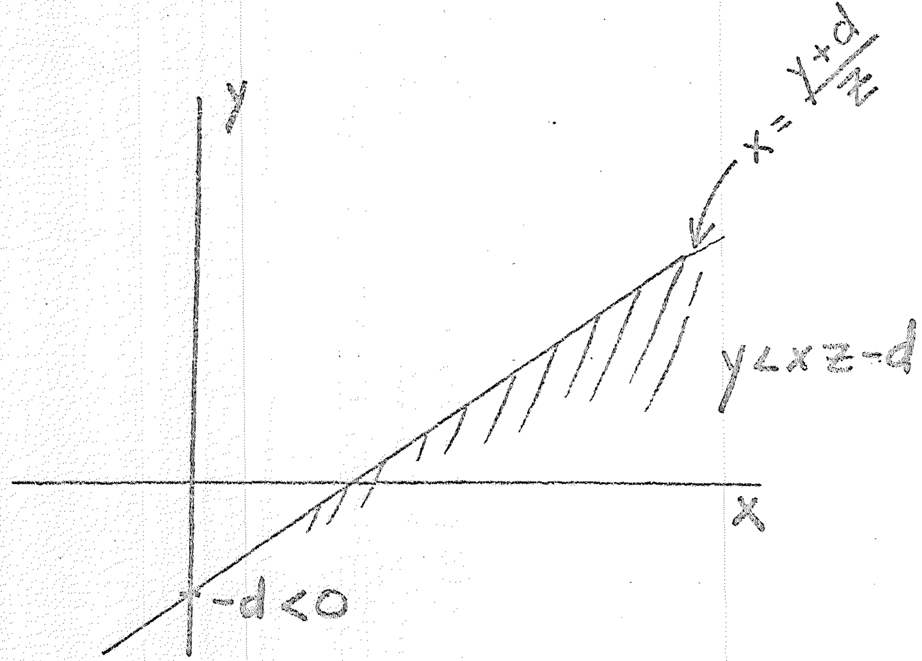
(f)

$$F_Z(z) = P_r[Z \le z]$$

$$= P_r\left[\frac{Y+d}{X} \le z\right]$$

$$= P_r[Y + d \le Xz]$$

$$= P_r[\;Y \le Xz - d]$$



$$F_Z(z) = \iint f_{XY}(x,y)\,dx\,dy$$

$$= \frac{ba^c}{\Gamma(c)} \int_{y=0}^{\infty} y^c e^{-ay} \int_{x=\frac{y+d}{z}}^{\infty} e^{-bxy}\,dx\,dy$$

$$= \frac{ba^c}{\Gamma(c)} \int_{y=0}^{\infty} y^c e^{-ay} \frac{1}{by} e^{-by\left(\frac{y+d}{z}\right)} dy$$

$$= \frac{a^c}{\Gamma(c)} \int_{y=0}^{\infty} y^{c-1} e^{-\left[\left(a+\frac{bd}{z}\right)y + b\frac{y^2}{z}\right]} dy$$

$$= \frac{a^c}{\Gamma(c)} \int_{y=0}^{\infty} y^{c-1} e^{-\frac{b}{z}\left[\left(\frac{az}{b}+d\right)y + y^2\right]} dy$$

$$= \frac{a^c}{\Gamma(c)} \int_{y=0}^{\infty} y^{c-1} e^{-\frac{b}{z}\left[y+\frac{1}{2}\left(\frac{az}{b}+d\right)\right]^2} dy\; e^{\frac{b}{4z}\left(\frac{az}{b}+d\right)^2}$$

For $c = 1$:

$$F_Z(z) = \frac{a^c}{\Gamma(c)} \int_{y=0}^{\infty} e^{-\frac{b}{z}\left(y+\frac{1}{2}\left(\frac{az}{b}+d\right)\right)^2} dy\; e^{\frac{b}{4z}\left(\frac{az}{b}+d\right)^2}$$

$$\frac{1}{2\sigma^2} = \frac{b}{z} \Longrightarrow \sigma = \sqrt{\frac{z}{2b}}$$

$$\text{Let } \psi = \frac{y + \frac{1}{2}\left(\frac{az}{b}+d\right)}{\sqrt{z/2b}}$$

-6-

Then:

$$F_z(z) = \frac{a^{1}}{\Gamma(1)} \int_{\psi = \frac{\frac{1}{z}\left(\frac{az}{b}+d\right)}{\sqrt{z/2b}}}^{\infty} e^{-\psi^2/2} \frac{d\psi}{\sqrt{\frac{z}{2b}}}\, e^{\frac{b}{4z}\left(\frac{az}{b}+d\right)^2}$$

$$= a\sqrt{\frac{2\pi}{z/2b}}\, e^{\frac{b}{4z}\left(\frac{az}{b}+d\right)^2} \left[\frac{1}{\sqrt{2\pi}} \int_{\psi = \frac{\frac{az}{b}+d}{\sqrt{z/2b}}}^{\infty} e^{-\psi^2/2}\, d\psi\right]$$

$$= 2a\sqrt{\frac{\pi b}{z}}\, e^{\frac{b}{4z}\left(\frac{az}{b}+d\right)^2} \left[\frac{1}{2} + \mathrm{erf}\left(\frac{az+bd}{\sqrt{zb/2}}\right)\right] u(z)$$

EE505   Final Examination   Summer '83

(name)

Score = ______________/175

1. A random variable X has a characteristic function:

$$\Phi_X(\omega) = A\cos^2(a\omega)$$

where a is a given parameter. Compute:

(a) A
(b) E(X)
(c) var(X)

Sometimes the expected value of a random variable is not such a good estimate. For example, let X be a Poisson random variable with parameter a=1. Let

$$Y = (-1)^X$$

Compute E(Y) and comment.

Let X be a gamma random variable with n an integer:

$$f_X(x) = \frac{c^{n+1}}{n\,!} x^n \exp(-cx)\, U(x)$$

Let $(X_n | n=1,2,\ldots,N)$ be iid samples from this density. Define

$$\overline{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$$

Find the density function for the average, $f_{\overline{X}}(x)$.

Let P and Q denote independent random variables both uniformly distributed on the interval from zero to unity. Define the process:

$$X(t) = P\, e^{-Qt}\, U(t)$$

Find:
(a) $E(X(t))$
(b) $R(t_1,t_2)$
(c) var $X(t)$

We draw N iid samples from a shifted Laplacian random variable with mean η and variance σ^2. Give an approximation of the density function for the average of these numbers. $N >> 1$

X(t) is a stationary random process with mean η and autocorrelation

$$R_X(\tau) = \overline{x^2} \ \exp \ -a|\tau|$$

where a is a specified parameter. What percentage of the time can we expect X(t) to lie below a given threshold, T?

In our take-home problem last week, we found that the joint density,

$$f_{XY}(x,y) = 8\, y^2\, e^{-2y}\, e^{-2xy}\, U(x)\, U(y)$$

had a marginal density

$$f_Y(y) = 4\, y\, e^{-2y}\, U(y).$$

Given that $Y=2$, what is a good estimate of X?

EE505 Final Examination Summer '83 Solutions!

1. A random variable X has a characteristic function:

$$\Phi_X(\omega) = A\cos^2(a\omega)$$

where a is a given parameter. Compute:

(a) A
(b) E(X)
(c) var(X)

(a) $\Phi_X(0) = 1 \Rightarrow A = 1$

(b) $\Phi_X(\omega) = \frac{1}{2}(1 + \cos 2a\omega)$

$$\frac{d\Phi_X}{d\omega} = -\frac{1}{2}\, 2a \sin 2a\omega = -a \sin 2a\omega$$

$$\frac{d\Phi_X(0)}{d\omega} = jE[X] = 0 \Rightarrow E(X) = 0$$

(c) $\frac{d^2\Phi_X}{d\omega^2} = -a(2a)\cos 2a\omega$

$$\frac{d^2\Phi_X(0)}{d\omega^2} = j^2 E(X^2) = -2a^2 \Rightarrow E(X^2) = 2a^2$$

$$\operatorname{var} X = E(X^2) - E^2(X)$$
$$= 2a^2$$

Sometimes the expected value of a random variable is not such a good estimate. For example, let X be a Poisson random variable with parameter a=1. Let

$$Y = (-1)^X$$

Compute E(Y) and comment.

$$E[Y] = \sum_{k=0}^{\infty} (-1)^k \frac{e^{-a} a^k}{k!} \quad ; a=1$$

$$= e^{-1} \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} = e^{-1} e^{-a}$$

$$= e^{-2}$$

$$= 0.135$$

COMMENT: Y is always ± 1

Let X be a gamma random variable with n an integer:

$$f_X(x) = \frac{c^{n+1}}{n\,!}\, x^n \exp(-cx)\ U(x)$$

Let $(X_n | n=1,2,\ldots,N)$ be iid samples from this density. Define

$$\overline{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$$

Find the density function for the average, $f_{\overline{X}}(x)$.

$$\Phi_{\overline{X}}(\omega) = E\left[e^{j\omega\overline{X}}\right] = E\left[e^{j\frac{\omega}{N}\sum_{n=1}^{N} X_n}\right]$$

$$= E\left[\prod_{n=1}^{N} e^{j\omega X_n/N}\right]$$

$$= \prod_{n=1}^{N} E\left[e^{j\omega X_n/N}\right]$$

$$= E^N\left[e^{j\omega X_n/N}\right]$$

$$= \Phi_X^N\left(\frac{\omega}{N}\right)$$

From p.154 of text:

$$\Phi_X(\omega) = \frac{c^{n+1}}{(c-j\omega)^{n+1}}$$

$$\Rightarrow \Phi_{\overline{X}}(\omega) = \frac{c^{N(n+1)}}{\left(c - j\frac{\omega}{N}\right)^{N(n+1)}} \quad ; \quad \hat{n}+1 = N(n+1) = \underbrace{(Nn+N+1)}_{\hat{n}}-1$$

From Fourier xform scaling theorem

$$f_{\overline{X}}(x) = N\ \frac{c^{\hat{n}+1}}{\hat{n}!}(Nx)^{\hat{n}}\, e^{-c(Nx)}\ \underline{U}(x)$$

$$= \frac{N\ c^{N(n+1)}}{(Nn+N+1)!}(Nx)^{Nn+N+1}\ e^{-cNx}\ \underline{U}(x)$$

Let P and Q denote independent random variables both uniformly distributed on the interval from zero to unity. Define the process:

$$X(t) = P\, e^{-Qt}\, U(t)$$

Find:
(a) $E(X(t))$
(b) $R(t_1,t_2)$
(c) var $X(t)$

(a) $E[X(t)] = E[Pe^{-Qt}U(t)]$

$= E[P]E[e^{-Qt}]u(t)$; $E[P] = \frac{1}{2}$

$E[e^{-Qt}] = \int_0^1 e^{-qt}dq = -\frac{1}{t}e^{-qt}\Big|_0^1 = \frac{1-e^{-t}}{t}$

$\Rightarrow E[X(t)] = \frac{1-e^{-t}}{2t}U(t)$

(b) $R(t_1,t_2) = E[X(t_1)X(t_2)] = E[P^2e^{-Q(t_1+t_2)}]U(t_1)U(t_2)$

$= E[P^2]\,E[e^{-Q(t_1+t_2)}]\,U(t_1)U(t_2)$

$E[P^2] = \int_0^1 p^2dp = \frac{1}{3}$

$E[e^{-Q(t_1+t_2)}] = \frac{1-e^{-(t_1+t_2)}}{t_1+t_2}$

$\Rightarrow R(t_1,t_2) = \frac{1-e^{-(t_1+t_2)}}{3(t_1+t_2)}U(t_1)U(t_2)$

(c) $E[X^2(t)] = R(t,t) = \frac{1-e^{-2t}}{6t}U(t)$

$\text{var}\, X(t) = E[X^2] - E(X)^2$

$= \left(\frac{1-e^{-2t}}{6t} - \left(\frac{1-e^{-t}}{2t}\right)^2\right)U(t)$

We draw N iid samples from a shifted Laplacian random variable with mean η and variance σ^2. Give an approximation of the density function for the average of these numbers. $N \gg 1$

Central limit theorem:

$$\overline{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$$

$$E[\overline{X}] = E[X] = \eta$$

$$\operatorname{var} \overline{X} = \frac{\sigma^2}{N}$$

$\Longrightarrow \overline{X}$ in normal (mean $= \eta$, variance $= \frac{\sigma^2}{N}$)

or

$$\overline{X} \sim \frac{\sqrt{N}}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\eta)^2}{2\sigma^2/N}}$$

X(t) is a stationary random process with mean η and autocorrelation

$$R_X(\tau) = \overline{x^2} \ \exp \ -a|\tau|$$

where a is a specified parameter. What percentage of the time can we expect X(t) to lie below a given threshold, T?

$$P_r[X(t) \leq T] = ?, \quad var = \sqrt{\overline{x}^2 - \eta^2}$$

$$X(t) \sim \frac{1}{\sqrt{2\pi}\sqrt{var}} e^{-\frac{(x-\eta)^2}{2var}}$$

$$P_r[X(t) \leq T] = \int_{-\infty}^{T} \frac{1}{\sqrt{2\pi}\, var} e^{-\frac{(x-\eta)^2}{2var}} dx$$

$$\text{Let} \quad y = \frac{x-\eta}{\sqrt{var}}$$

$$P_r[X(t) \leq T] = \int_{-\infty}^{\frac{T-\eta}{\sqrt{var}}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$= \frac{1}{2} + erf \frac{T-\eta}{\sqrt{var}}$$

$$= \frac{1}{2} + erf\left(\frac{T-\eta}{\sqrt{\overline{x^2} - \eta^2}}\right)$$

Solutions

EE505 Final Examination

Robert J. Marks II

August 20, 1997; 2:20 to 4:20 PM

The examination is closed book and closed notes. No calculators are allowed. Each student is allowed three sheets of notes. All problems are weighted equally. Work must be done in ink.

All work will be done in a test booklet. No scratch paper is needed.

"And I trust that you will discover that we have not failed the test.", 2 Corinthians 13:6 (English-NIV)

1. Let X and Y be independent random variables and let $Z = X + Y$. Prove or disprove the following general propositions.
   (a) $\bar{Z} = \bar{X} + \bar{Y}$
   (b) $\overline{Z^2} = \overline{X^2} + \overline{Y^2}$
   (c) $\text{var}(Z) = \text{var}(X) + \text{var}(Y)$.
   (d) $\text{var}(aZ) = a^2 \,\text{var}Z$.

$\overline{Z} = \overline{X+Y} = \overline{X} + \overline{Y} \Leftarrow$ TRUE

$$\overline{Z^2} = \overline{(X+Y)^2} = \overline{X^2} + 2\overline{X}\,\overline{Y} + \overline{Y^2} \neq \overline{X^2} + \overline{Y^2}$$

(True if $\overline{X}\,\overline{Y} = 0$, but not in general).

$$\Phi_Z(\omega) = \Phi_X(\omega)\,\Phi_Y(\omega) \Leftarrow \text{if Ind.}$$

$$\Psi_Z(\omega) = \Psi_X(\omega) + \Psi_Y(\omega)$$

$$\Psi_Z''(0) = \Psi_X''(0) + \Psi_Y''(0)$$

$$\Rightarrow \text{var}\,Z = \text{var}\,X + \text{var}\,Y$$

$$\text{var}(aZ) = \overline{(aZ)^2} - (\overline{aZ})^2 = a^2\overline{Z^2} - a^2\overline{Z}^2 = a^2\,\text{var}\,Z$$

True!

2.

$$Y = \frac{1}{N}\sum_{k=1}^{N} X_k^2$$

where the X_k's are i.i.d. random variables with probability density function

$$f_X(x) = e^{-x}U(x)$$

Estimate the probability density function for the random variable Y when N is large.[1]

$$Z_k = X_k^2$$

$$\overline{Z_k} = \overline{X_k^2} = 2! = 2$$

$$\overline{Z_k^2} = \overline{X_k^4} = 4! = 24$$

$$\Rightarrow \text{var } Z_k = 24 - 4 = 20$$

$$W = \sum_{k=0}^{N} Z_k \Rightarrow \overline{W} = N\,\overline{Z_k} = 2N$$

$$\text{var } W = N \text{ var } Z = 20N$$

By C.L.T. $W \sim n(2N, \sqrt{20N})$

$$Y = \frac{W}{N} \sim n\left(2, \sqrt{\frac{\text{var } W}{N^2}}\right)$$

From Prob. 1d

$$= n\left(2, \sqrt{\frac{20}{N}}\right)$$

$$= \frac{1}{\sqrt{2\pi \frac{20}{N}}} e^{-\frac{(y-2)^2}{2(20/N)}}$$

$$= \sqrt{\frac{N}{40\pi}}\, e^{-\frac{N(y-2)^2}{40}}$$

---

[1] Recall from the last test that the nth moment of each X_k is $(\cancel{n-1})!$. $n!$

3. A total of N i.i.d. Bernoulli trials with probability of success p are performed. The outcome of trial m, the random variable X_m, is set to one if there is a success and zero otherwise. We form the sum

$$Y = \sum_{m=1}^{N} X_m.$$

Evaluate the exact probability density function for the random variable Y.

This is a disguised binomial R.V.

$$p_k = \Pr[Y = k] = \binom{N}{k} p^k q^{N-k} \; ; \; q = 1-p$$

$$f_Y(x) = \sum_k p_k \, \delta(x-k)$$

$$= \sum_{k=0}^{N} \binom{N}{k} p^k q^{N-1} \, \delta(x-k)$$

4. The Weibull random variable Y with positive parameters A and B is

$$F_Y(y) = \left[1 - \exp\left\{-\left(\frac{y}{A}\right)^B\right\}\right] U(y).$$

Let X be a uniform random variable on the interval $(0,1)$. Given A and B, find a random variable transformation, $Y = g(X)$, to produce a Weibull random variable.

$$x = 1 - e^{-\left(\frac{y}{A}\right)^B}$$

$$e^{-\left(\frac{y}{A}\right)^B} = 1 - x$$

$$\left(\frac{y}{A}\right)^B = -\ln(1-x)$$

$$y = A\left[-\ln(1-x)\right]^{1/B}$$

Thus

$$Y = g(X)$$

where

$$g(x) = A\left[-\ln(1-x)\right]^{1/B}$$

This is a strictly increasing solution. We can show a strictly decreasing solution:

$$g(x) = A\left[-\ln(x)\right]^{1/B}.$$

Are the other $g(x)$'s that will work? Of course!

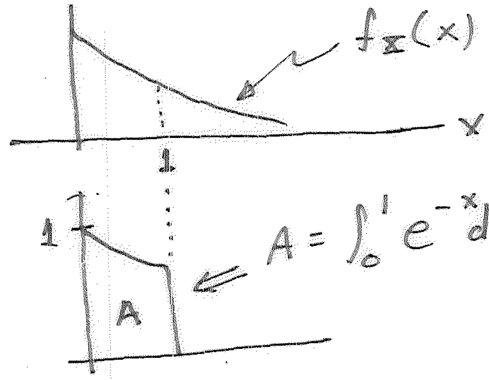
5. A random variable has a probability density function of

$$f_X(x) = e^{-x}U(x)$$

We take i.i.d. samples from this distribution until we get a number between zero and one - and then stop. Call this last random variable Y. Evaluate the probability density function of Y.

~~$f_Y(y) = \Pr[X \le$~~

$$f_Y(x) = f_X(x \mid 0 \le X \le 1)$$



$$A = \int_0^1 e^{-x}dx = e^{-x}\Big|_1^0 = 1 - e^{-1}$$

Thus

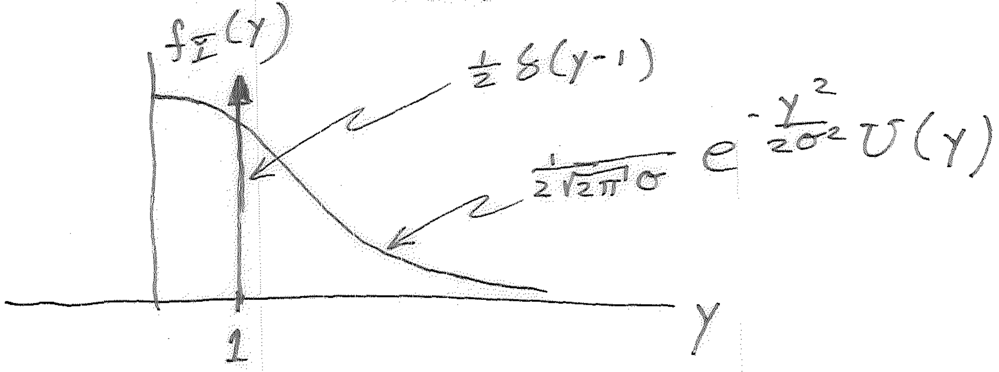
$$f_Y(x) = \begin{cases} \dfrac{e^{-x}}{1-e^{-1}} & ;\ 0 \le x \le 1 \\ 0 & ;\ \text{otherwise} \end{cases}$$

6. Let X be a zero mean normal random variable with variance σ^2. Let $Y = X$ when X is positive and let $Y = 1$ otherwise. Evaluate and sketch the probability density function for Y.

By inspection:

$$f_Y(y) = \frac{1}{2}\delta(y-1) + \frac{1}{2} f_X(y)\,U(y)$$

$$= \frac{1}{2}\delta(y-1) + \frac{1}{2\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}} U(y)$$

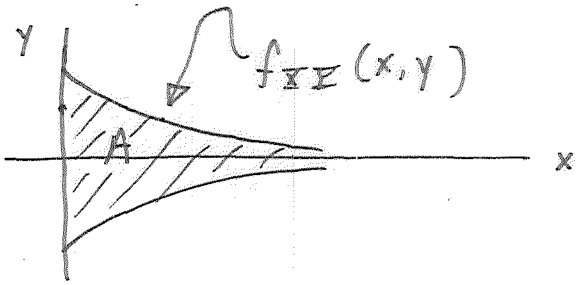


7. A joint probability density function is defined by

$$f_{XY}(x,y) = \begin{cases} A & ; |y| \le e^{-x} \text{ and } \cancel{y}\, x \ge 0 \\ 0 & ; \text{otherwise} \end{cases}$$

(a) Evaluate A.

(b) Evaluate the marginal ~~distribution,~~ density $f_Y(y)$.



(a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

$$= A\int_{x=0}^{\infty}\int_{y=-e^{-x}}^{e^{-x}} dy\,dx = A\int_{x=0}^{\infty} 2e^{-x}\,dx = 2A$$

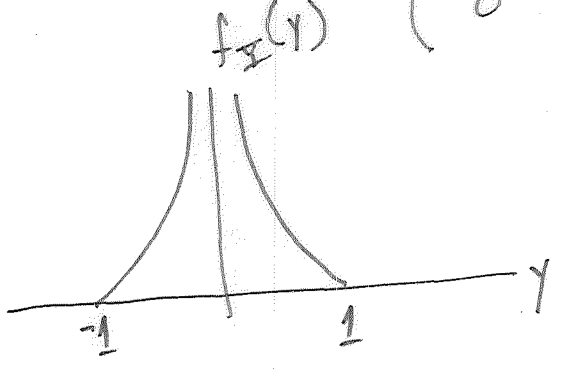
$$\Rightarrow A = \frac{1}{2}$$

$y = \pm e^{-x}$

$+x = \pm \ln y$

(b) $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

$$= \frac{1}{2}\int_{0}^{-\ln|y|} dx = \begin{cases} -\frac{1}{2}\ln|y| & ; |y| < 1 \\ 0 & ; \text{otherwise} \end{cases}$$



8. Does there exist a zero mean random variable where the the Tchebycheff inequality is met? In other words, is there a probability density function for which, for all $k \ge 1$,

$$P\{|X| \ge k\sigma\} = \frac{1}{k^2} \quad ?$$

If so, please specify $F_X(x)$. If not, please explain why.

There are a number of ways to show the answer is "no".

Way #1: $\Pr[|X| \ge k\sigma] = \frac{1}{k^2} = 1-(F_X(k\sigma) - F_X(-k\sigma))$; $k \ge 1$

Differentiate wrt k $\quad -\frac{2}{k^3} = -\sigma f_X(k\sigma) - \sigma f_X(-k\sigma)$; $k \ge 1$

$$\Rightarrow f_X(k\sigma) + f_X(-k\sigma) = \frac{+2}{\sigma k^3} \quad ; k \ge 1$$

set $x = k\sigma$ $\quad \Rightarrow f_X(x) + f_X(-x) = \frac{2\sigma^2}{x^3}$; ~~X~~ $x \ge \sigma$

since $\overline{X} = 0$, $\overline{X^2} = \text{var}\, X = \sigma^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2 [f_X(x) + f_X(-x)]\,dx$$

$$\ge \frac{1}{2}\int_{x=\sigma}^{\infty} x^2 [f_X(x) + f_X(-x)]\,dx$$

$$= \frac{1}{2}\int_{x=\sigma}^{\infty} x^2 \, \frac{2\sigma^2}{x^3}\,dx = \frac{\sigma^2}{2}\int_{x=\sigma}^{\infty}\frac{dx}{x}$$

$$= \frac{\sigma^2}{2} \ln x \Big|_{\sigma}^{\infty} = \infty \Rightarrow$$ violating assumption $\text{var}\, X = \sigma < \infty$.

Way #2 We can rewrite

$$\Pr[|X| \ge \varepsilon] \le \frac{\sigma^2}{\varepsilon} \Leftarrow$$ Tchebycheff's inequality

This is the way it is on p. 114 of the text. In the derivation, for integration over the interval $|x| \ge \varepsilon$, we use the inequality $x^2 \ge \varepsilon^2$. From 3rd equation on p. 114,

$$\sigma^2 \ge \int_{|x| \ge \varepsilon} x^2 f_X(x)\,dx \overset{Ⓐ}{\ge} \varepsilon^2 \int_{|x| > \varepsilon} f_X(x)\,dx$$

Since x^2 is changing and ε^2 is not, the inequality Ⓐ should be a strict inequality. Thus, the bound is never met.

There are other ways

EE 505

Midterm
Wednesday, July 16, 1997
2:20 PM to 4:30

Solutions

INSTRUCTIONS:

- Rip off the last sheet of paper in this test booklet and put it in your pocket. It is your homework assignment due in class one week from today.
- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed a single sheet of notes.
- No calculators pleased.
- Please use an ink pen.
- Each problem is worth 20 points.
- TIE Students: Please ask your proctor to write his/her name on the cover, indicate the times over which the exam was administered and initial.

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1. Specify whether the following pairs of events are (a) independent, (b) mututally exclusive (c) both or (d) neither. Two points will be given for a correct answer, zero for no answer and -1 for an incorrect answer.

- You pass this test. You fail this test. (You may assume the probability of both events is nonzero). Mutually Exclusive
- A rolled die shows three dots. A flipped coin shows heads.
  Independent
- The sum on two dice is seven. There are six dots on the first die. B
  A Independent $P_r[A \cdot B] = 1/36$ $P[A] = P[B] = 1/6$
- The sum on two dice is seven. There are six dots on one of the two dice.
  neither B $\Rightarrow P_r[B] = 11/36$
- You have an ace in your poker hand. Your opponent has an ace in their poker hand.
  neither
- You have an ace in your poker hand. Your opponent has the king of hearts in their poker hand.
  neither
- You have the king of hearts in your poker hand. Your opponent has the king of hearts in their poker hand.
  mutualy exclusive
- You win the Washington state lottery. Your mother wins the Ohio State lottery. (Both of you purchased tickets.)
  Ind.
- You receive one telephone call before noon. You receive two calls all day.
  neither
- You roll two conventional six sided dice. The first die shows three dots. The second die shows thirty eight dots.
  both B $\Rightarrow P_r[B] = P_r[\phi] = 0$

2. Ken Griffey Jr. has a batting average of 0.333. Assume this means, each time he bats, his probability of getting a hit is 1/3. Estimate the probability that he gets between $\overset{k_1}{\overset{\shortparallel}{20}}$ and $\overset{k_2}{\overset{\shortparallel}{28}}$ hits (inclusive) in his next 72 at bats.

$$P_r[k_1 \le k \le k_2] \simeq G\left(\frac{k_2-np}{\sqrt{npq}}\right) - G\left(\frac{k_1-np}{\sqrt{npq}}\right)$$

$$\sqrt{npq} = \sqrt{72\times\frac{1}{3}\times\frac{2}{3}} = \sqrt{16} = 4$$

$$np = \frac{1}{3}\times 72 = 24$$

$$\frac{k_2-np}{\sqrt{npq}} = \frac{28-24}{4} = 1$$

$$\frac{k_1-np}{\sqrt{npq}} = \frac{20-24}{4} = -1$$

$$\Rightarrow P_r[k_1 \le k \le k_2] \simeq \mathrm{erf}(1) - \mathrm{erf}(-1)$$

$$= 2\,\mathrm{erf}(1)$$

$$= 2\times 0.34$$

$$= 0.68$$

3. In the HUB, there are 10 AA, 20 civil engineering and 30 EE students eating husky burgers. Five students are chosen at random. What is the probability that there are exactly three EE's and exactly one civil engineering student chosen?

If you assume choice with replacement:

$$p_1 = \frac{10}{60} = \frac{1}{6} \quad , p_2 = \frac{20}{60} = \frac{1}{3} \quad , p_3 = \frac{30}{60} = \frac{1}{2} \; , n = 5$$

$$\Pr[k_1 , k_2, k_3] = \frac{n!}{k_1!\, k_2!\, k_3!}\, p_1^{k_1}\, p_2^{k_2}\, p_3^{k_3}$$

$$\Pr[k_1 = 1, k_2 = 1, k_3 = 3] = \frac{5!}{1!1!\ 3!}\ \frac{1}{6} \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3$$

$$= 5 \cdot \cancel{4} \times \frac{1}{18 \times \cancel{4} \times 2} = 5/36$$

If you assume no replacement, it's pretty rough to figure

4. Bill eats only sushi and sausage pizzas. Sushi give him heartburn 10% of the time. The pizza's give him heartburn 20% of the time. He eats twice as many pizzas as sushi. Bill has heartburn. What is the probability it was caused by a sausage pizza?

$$2P_{PIZZA} = P_{SUSHI} \Rightarrow P_{PIZZA} = 2/3 \quad , P_{sushi} = 1/3$$

H = heartburn

$$Pr[PIZZA/H] = \frac{Pr[H/PIZZA]\,Pr[PIZZA]}{Pr[H/PIZZA]\,Pr[PIZZA] + Pr[H/sushi]\,Pr[sushi]}$$

$$= \frac{\frac{1}{5} \times \frac{2}{3}}{\frac{1}{5} \times \frac{2}{3} + \frac{1}{10} \times \frac{1}{3}} = \frac{2/15}{4/30 + 3/30}$$

$$= \frac{2/15}{7/30} = \frac{4/30}{7/30} = 4/7$$

5. Poisson points with parameter $\lambda = 2$ occuraces per hour are observed for a half hour. What is the probability that the number of occurances exceeds two *given* that the total number of occurances exceeds one?

$\lambda = 2/\text{HR}$

$$P_r[X > 2 \mid X \geq 1] = \frac{P_r[X > 2, X > 1]}{P_r[X > 1]} = \frac{P_r[X > 2]}{P_r[X > 1]}$$

$$= \frac{1 - P_r[X \leq 2]}{1 - P_r[X \leq 1]} = \frac{1 - e^{-a}\left[\frac{a^0}{0!} + \frac{a^1}{1!} + \frac{a^2}{2!}\right]}{1 - e^{-a}\left[\frac{a^0}{0!} + \frac{a^1}{1!}\right]}; \quad a = 2 \times \tfrac{1}{2} = 1$$

$$= \frac{1 - e^{-1}[1 + 1 + 2]}{1 - e^{-1}[1 + 1]} = \underset{\uparrow}{\frac{1 - 4e^{-1}}{1 - 2e^{-1}}} = 0.560$$

CAN LEAVE
ANSWER THIS WAY.

6. Consider a Bernoulli trial with probability of success p. We perform the Bernoulli trial until we get a success. Let M denote the number of trials needed to achieve a success.

1. What is the probability that, for a given positive integer, m, that $M = m$?
2. Do all of the probabilities add to one?[1]

1.

m	$P_r[M=m]$
1	p
2	qp
3	q^2p
4	q^3p
$\vdots$	
m	$q^{m-1}p$

$$q^{m-1}p \Rightarrow P_r[M=m] = \begin{cases} q^{m-1}p & ; m=1,2,\ldots \\ 0 & ; \text{otherwise} \end{cases}$$

2. Yes. For credit, show this:

$$p\sum_{m=1}^{\infty} q^{m-1} \underset{\substack{\uparrow \\ n=m-1}}{=} p\sum_{n=0}^{\infty} q^n \underset{\substack{\uparrow \\ \text{From} \\ \text{"Hint"}}}{=} \frac{p}{1-q} = \frac{p}{p} = 1$$

---

[1] Recall the geometric series where, if $|a| < 1$,

$$\sum_{m=0}^{\infty} a^m = \frac{1}{1-a}.$$

1. SCRATCH PAPER

Homework #3
Due in class on July 28.

1. From Papoulis, Chapter 4: Problems 4,6,7,8,9,10,12,13,14,17,18,19,21

2. There are two classes of objects - a right group and a left group. Both are distributed as Gaussian random variables. The left class has a mean of p_L and a standard deviation of σ_L. The right class has parameters p_R and σ_R. An element is chosen with equal probability from one of the goups and the result of the experimaent is $\mathbf{P} = p$. Show that

$$\text{Probability[the element is from the left class } |\mathbf{P} = p] = \frac{1}{1 + \frac{\sigma_L}{\sigma_R} \exp\left[\frac{1}{2}\left(\frac{\eta_L}{\sigma_L^2} - \frac{\eta_R}{\sigma_R^2}\right)\right]}$$

where

$$\eta_L = (p - p_L)^2$$

and

$$\eta_R = (p - p_R)^2.$$

$$P_r[P=p] = \frac{\frac{1}{\sqrt{2\pi}\,\sigma_L} e^{-\frac{(p-p_L)^2}{2\sigma_L^2}} \cancel{P_r[P_L]}}{\frac{1}{\sqrt{2\pi}\,\sigma_L} e^{-\frac{(p-p_L)^2}{2\sigma_L^2}} \cancel{P_r[P_L]} + \frac{1}{\sqrt{2\pi}\,\sigma_R} e^{-\frac{(p-p_R)^2}{2\sigma_R^2}} \cancel{P_r[P_R]}}$$

$$= \frac{1}{1 + \frac{\sigma_L}{\sigma_R} e^{\frac{1}{2}\left[\frac{\eta_L}{\sigma_L^2} - \frac{\eta_R}{\sigma_R^2}\right]}}$$

EE 505
Midterm
Monday, November 24, 1997
1:30 PM to 3:20

Solutions

INSTRUCTIONS:

- Look on the web for the next homework assignment due one week from today.
- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed two sheets of notes.
- No calculators pleased.
- Please use an ink pen.
- Each problem is worth 20 points.
- TIE Students: Please ask your proctor to write his/her name on the cover, indicate the times over which the exam was administered and initial.

Some potentially helpful equations follow.

$$\sum_{k=1}^{\infty} \frac{1}{n} = \infty$$

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

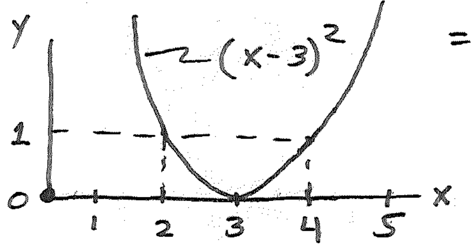
$$\sin(\theta) = \frac{1}{j2}\left(e^{j\theta} - e^{-j\theta}\right)$$

---

1. Let X be a Poisson random variable with parameter $a = 1$. Let $Y = (X-3)^2$. Then

$$\text{Probability } [-1.5 \le Y < 1.2] = c \times e^{-1}.$$

Evaluate the constant c.



$$= P_r[Y = 0 \text{ or } Y = 1]$$

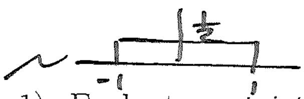
$$= P_r[X = 3 \text{ or } X = 2 \text{ or } X = 4]$$

$$= e^{-a}\left(\frac{a^3}{3!} + \frac{a^2}{2!} + \frac{a^4}{4!}\right)$$

$$= e^{-1}\left(\frac{1}{6} + \frac{1}{2} + \frac{1}{24}\right)$$

$$= e^{-1}\left(\frac{4+12+1}{24}\right) = e^{-1}\left(\frac{17}{24}\right)$$

$$\Rightarrow c = 17/24$$



2. Let X be a uniform random variable on the interval $(-1, 1)$. Evaluate a *strictly decreasing* nonlinearity, $g(x)$ so that the random variable $Y = g(X)$ is distributed as

$$f_Y(z) = 2e^{-2z}U(z)$$

where $U(\cdot)$ is the unit step.

$$f_Y(y) = \underbrace{\left|\frac{dg^{-1}}{dy}\right| f_X(g^{-1}(y))} = 2e^{-2y}U(y)$$

$$\frac{1}{2}\left|\frac{dg^{-1}}{dy}\right| = 2e^{-2y} \Rightarrow \frac{dg^{-1}}{dy} = \pm 4e^{-2y}$$

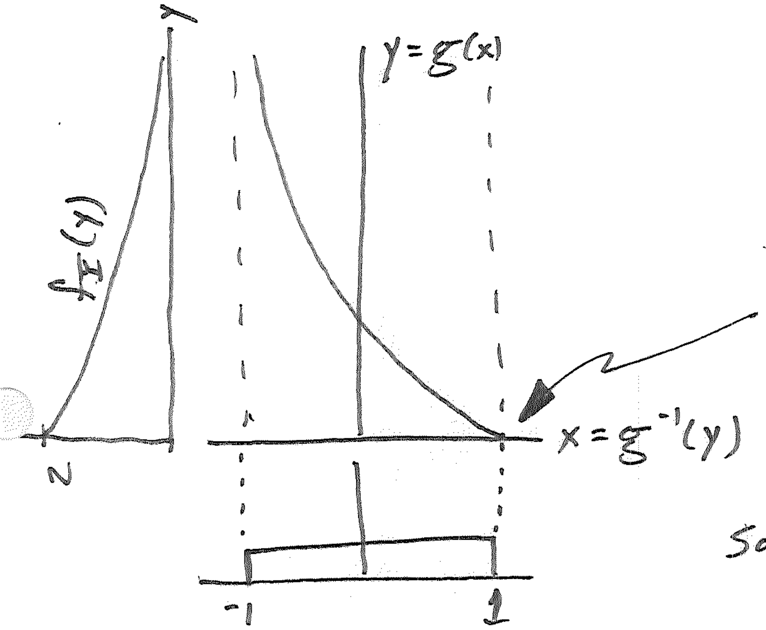
$$g^{-1}(y) = x = \mp 2e^{-2y} + c$$

choose + for strictly decreasing

$$g^{-1}(y) = 2e^{-2y} + c = x$$

$$x = 1 \Rightarrow y = 0$$

$$\Rightarrow 2 + c = 1 \Rightarrow c = -1$$



$$g^{-1}(y) = 2e^{-2y} - 1 = x$$

Solve for y:

$$2e^{-2y} = x + 1$$

$$e^{-2y} = \frac{x+1}{2} \Rightarrow -2y = \ln\frac{x+1}{2}$$

or $y = -\frac{1}{2}\ln\frac{x+1}{2} = g(x)$

3. Can the function $\cos(\alpha\omega)$ be a characteristic function? If not, why? If so, what is the corresponding probability density function?

Yes

Note:

$$\delta(x-a) \longleftrightarrow e^{j\omega a}$$

Thus, since

$$\cos \alpha\omega = \frac{1}{2}\left[e^{j\alpha\omega} + e^{-j\alpha\omega}\right]$$

$$\Longrightarrow f_X(x) = \frac{1}{2}\delta(x-\alpha) + \frac{1}{2}\delta(x+\alpha) \longleftrightarrow \cos\alpha\omega = \Phi_X(\omega)$$

4. Let X be a uniform random variable on the interval $(0,1)$. Find the value of the constant a so that

$$E\left(3(X+a)^2\right) = 1.$$

$$\overline{X^n} = \int_0^1 x^n dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$$

$$\overline{3(X+a)^2} = 3(\overline{X^2} + 2a\overline{X} + a^2)$$

$$= 3\left(\frac{1}{3} + \cancel{2}a\frac{1}{\cancel{2}} + a^2\right)$$

$$= 1 + 3a + 3a^2 = 1$$

$$\Rightarrow 3a + 3a^2 = 0$$

$$1 + a = 0 \Rightarrow a = -1$$

<u>or</u>, simply integrate

$$3\overline{(X+a)^2} = 3\int_0^1 (x+a)^2 dx = 1$$

$$\downarrow$$

5. Express

$$E[cos(2\pi Xt]$$

in terms of real part of the characteristic function of X.

$$\overline{\cos 2\pi X t} = \frac{1}{2}\,\overline{e^{j2\pi X t}} + \frac{1}{2}\,\overline{e^{-j2\pi X t}}$$

$$= \frac{1}{2}\Phi_X(2\pi t) + \frac{1}{2}\Phi_X(-2\pi t)$$

But:

$$\Phi_X(-2\pi t) = \int_{-\infty}^{\infty} f_X(x)\, e^{-j2\pi\omega x}\, dx$$

$$= \left[\int_{-\infty}^{\infty} f_X(x)\, e^{j2\pi\omega x}\, dx\right]^*$$

$$= \Phi_X^*(2\pi t)$$

$$\Rightarrow \overline{\cos 2\pi X t} = \frac{1}{2}\Phi_X(2\pi t) + \frac{1}{2}\Phi_X^*(2\pi t)$$

$$= \mathrm{Re}\,\Phi_X(2\pi t)$$

6. Is the following inequality true when the probability density function of X is zero for negative x?

$$\text{Probability}[X \geq a] \leq E\left[\left(\frac{X}{a}\right)^{2n}\right]$$

If so, please show. If not, give a counterexample.

Sure.

Markoff Inequality

$$\Pr[Y \ge \alpha] \le \overline{Y}/\alpha$$

$$Y = X^{2n}$$

$$\Rightarrow \Pr[X^{2n} \ge \alpha] \le \frac{\overline{X^{2n}}}{\alpha}$$

$$= \Pr[X \ge \alpha^{\frac{1}{2n}}]$$

set $a = \alpha^{\frac{1}{2n}}$. This gives

$$\Pr[X \ge a] \le \frac{\overline{X^{2n}}}{a^{2n}}$$

EE 505
Midterm

INSTRUCTIONS:

- Monday, July 15, 1996; 2:20 PM to 4:20 PM.
- Write your name on the upper right hand side of this sheet.
- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed a single legal sized sheet of notes and calculator.
- Each problem is worth 20 points.
- TIE students must identify the exam proctor and have the proctor initial the examination.

1. Specify whether the following pairs of events are (a) independent, (b) mututally exclusive or (c) neither[1].

- A balanced budget amendment bill passes congress by August 15. A balanced budget amendment does not pass congress by August 15.
- The sum on two dice is seven. There are six dots on the first die.
- You have an ace in your poker hand. Your opponent has an ace in their poker hand.
- You win the Washington state lottery. Your mother wins the New York lottery.
- You receive one call before noon. You receive two calls all day.

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[1] Four points for a correct answer, zero for no answer and -2 for an incorrect answer.

2. Ken Griffey Jr. has a batting average of 0.300. Assume this means, each time he bats, his probability of getting a hit is 0.300. Estimate the probability that he gets 300 or more hits in his next 1000 at bats.

3. In Lake Wobegon, there are 10,000 catfish, 20,000 perch and 30,000 dogfish. The dogfish are very hungry and are twice as likely to be caught. Bobby Jack went fishing and caught four fish. The probability that three were dogfish and one was a perch were caught can be written as

$$\frac{2^P}{3^Q}.$$

What are the integers P and Q?

4. Bill eats only Big Macs and sausage pizzas. Bic Macs give him heartburn 10% of the time. The pizza's give him heartburn 20% of the time. He eats twice as many pizzas as Big Macs. Bill has heartburn. What is the probability it was caused by a Big Mac?

5. A Poisson random variable with parameter $\lambda = 2$ occurances per hour is observed for a half hour. The probability that the number of occurances exceeds or is equal to two *given* that the total number of occurances exceeds or equals one can be written as

$$\frac{1 - a\ \mathrm{e}^{c}}{1 - b\ \mathrm{e}^{d}}.$$

Identify the numbers a, b, c and d.

6. Washington state apples are modeled with a Gaussian pdf. If X is the diameter,

$$X \sim N(\mu = 3, \sigma = 2)$$

Apples below two inches in diameter and above four inches are discarded. What is the probability that an apple passing this test is three inches or less in diameter?

7. Matlab's error function is

$$\mathrm{erf}_{ML}(x) = \frac{2}{\sqrt{\pi}} \int_{t=0}^{x} \mathrm{e}^{-t^2} dt$$

Papoulis' definition is

$$\mathrm{erf}(y) = \frac{1}{\sqrt{2\pi}} \int_{z=0}^{y} \mathrm{e}^{\frac{-z^2}{2}} dz$$

We wish to find erf(2) using Matlab. How do you do it?

Find Scratch Paper #1

Scratch Paper #2

Scratch Paper #3

Scratch Paper #4

Scratch Paper #5

Table 3-1 $\operatorname{erf} x = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^x e^{-y^2/2}\, dy = \mathbb{G}(x) - \dfrac{1}{2}$

x	erf x	x	erf x	x	erf x	x	erf x
0.05	0.01994	0.80	0.28814	1.55	0.43943	2.30	0.48928
0.10	0.03983	0.85	0.30234	1.60	0.44520	2.35	0.49061
0.15	0.05962	0.90	0.31594	1.65	0.45053	2.40	0.49180
0.20	0.07926	0.95	0.32894	1.70	0.45543	2.45	0.49286
0.25	0.09871	1.00	0.34134	1.75	0.45994	2.50	0.49379
0.30	0.11791	1.05	0.35314	1.80	0.46407	2.55	0.49461
0.35	0.13683	1.10	0.36433	1.85	0.46784	2.60	0.49534
0.40	0.15542	1.15	0.37493	1.90	0.47128	2.65	0.49597
0.45	0.17364	1.20	0.38493	1.95	0.47441	2.70	0.49653
0.50	0.19146	1.25	0.39435	2.00	0.47726	2.75	0.49702
0.55	0.20884	1.30	0.40320	2.05	0.47982	2.80	0.49744
0.60	0.22575	1.35	0.41149	2.10	0.48214	2.85	0.49781
0.65	0.24215	1.40	0.41924	2.15	0.48422	2.90	0.49813
0.70	0.25804	1.45	0.42647	2.20	0.48610	2.95	0.49841
0.75	0.27337	1.50	0.43319	2.25	0.48778	3.00	0.49865

Papoulis
2nd ed
H.W.

#1: Chapt 2: 1, 8, 12, 14, 17, 19, 21

#2: Chapt 3: 2, 7, 10, 13, Chapt 4: 1, 2, 6, 10, 11

#3: Chapt 3: 3 Chapt 5: 1, 3, 5, 6, 10, 12, [redacted], [redacted], [redacted]

#4: Chapt 5: 14, 21, 23, 27*, 28*, 30
Chapt 6: 2, 3, 5

#5: Chapt 6: 9, 11, 14

#6: Chapt 7: [redacted], 10, 12, 15
" 8: 14, 19, 23, 24, 25, 26

#7: Chapt 9: 1, 2, 3, 4, 5, 6, 12, 14

#8: Chapt 9: 7, 11, 16, 21, 25, [redacted]
Chapt 10: 3, 5, 7,

2 MT'S	25
1 FINAL	35
H.W.	15

#1 Chapt 2: 1, 2, 3, 4, 12, 13

$$S-B \leq B \leq S$$

$$S \leq 2B \leq B+S$$

$$\frac{(}{\binom{10}{7}}$$